7. (i) Prove by induction that, for $n \in \mathbb{N}$,

$$\begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}^n = \begin{pmatrix} 3n+1 & -n \\ 9n & 1-3n \end{pmatrix}$$

(6)

(ii) Consider the statement

$$n^2 < 2^n \qquad \text{for all } n \in \mathbb{Z}^+$$

A student attempts to prove this statement using induction as follows.

Student's response

For $n = 1$ we have $1^2 = 1$ and $2^1 = 2$
Since $1 < 2$ the statement is true for $n = 1$

Suppose it is true for $n = k$, so $k^2 < 2^k$

Line 4 → Then $(k+1)^2 = k^2 + 2k + 1 < k^2 + k^2$ (since $2k + 1 < k^2$ for $k \in \mathbb{Z}^+$)

$= 2k^2$

$< 2 \times 2^k$ (by the assumption $k^2 < 2^k$)

$= 2^{k+1}$

Hence the result is true for $n = k + 1$

So the result is true for $n = 1$ and if it is true for $n = k$ then it is true for $n = k + 1$, and hence it is true for all positive integers n by mathematical induction.

(a) Show by a counterexample that the statement is not true.

Given that the only mathematical error in the student's proof occurs in line 4,

(b) identify the error made in the student's proof,

(c) hence determine for which positive integers the statement is true, explaining your reasoning.

(5)