7.

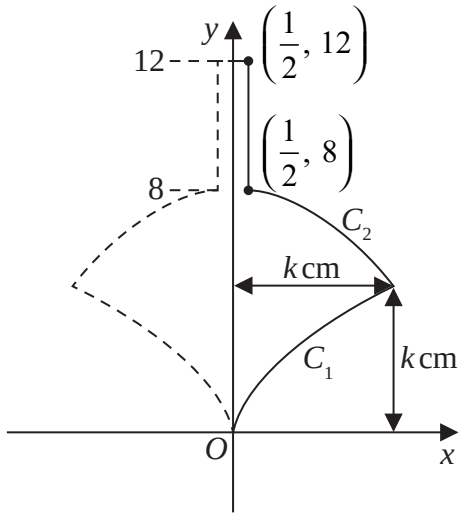

Diagram not to scale

Figure 2

Figure 2 shows a sketch of the cross-section of a design for a child's spinning top. The top is formed by rotating the region bounded by the y-axis, the curve C_1, the curve C_2, the line with equation $x = \frac{1}{2}$ and the line with equation $y = 12$, through 360° about the y-axis.

The curve C_1 has equation

$$y = k^{\frac{2}{3}}x^{\frac{1}{3}} \qquad 0 \leqslant x \leqslant k$$

and the curve C_2 has equation

$$y = \frac{32k^2 - k - (32 - 4k)x^2}{4k^2 - 1} \qquad \frac{1}{2} \leqslant x \leqslant k$$

(a) Show that $\int_k^8 \left((4k^2 - 1)y - (32k^2 - k)\right)\mathrm{d}y = \frac{1}{2}(8 - k)\left(4k^3 - 32k^2 + k - 8\right)$

(3)

Hence find

(b) the value of k that gives the maximum value for the volume of the spinning top,

(9)

(c) the maximum volume of the spinning top.

(3)