

7. Relative to a fixed origin O

- the point A has position vector $8\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$
- the point B has position vector $t\mathbf{i} + 2t\mathbf{j} + 5t\mathbf{k}$

where t is a non-zero constant.

(a) Show that $|\vec{AB}|^2 = 30t^2 - 24t + 77$

$$\begin{aligned} \text{(a)} \quad \vec{AB} &= \vec{OB} - \vec{OA} \\ &= \begin{pmatrix} t \\ 2t \\ 5t \end{pmatrix} - \begin{pmatrix} 8 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} t-8 \\ 2t+3 \\ 5t-2 \end{pmatrix} \quad (1 \text{ mark}) \end{aligned} \quad (3)$$

(b) Hence find

- the value of t when $|\vec{AB}|$ takes its minimum value,
- the minimum value of $|\vec{AB}|$, giving your answer as a simplified surd.

$$\text{(a) contd} \quad |\vec{AB}|^2 = (t-8)^2 + (2t+3)^2 + (5t-2)^2 \quad (1 \text{ mark})$$

Given that

- $|\vec{AB}|$ takes its minimum value $= t^2 - 16t + 64 + 4t^2 + 12t + 9 + 25t^2 - 20t + 4$
- the point C lies on AB such that C divides AB in the ratio $5:3$ $= 30t^2 - 24t + 77$ (1 mark)

(c) find the coordinates of C

(2)

(b) Let $y = |\vec{AB}|^2 = 30t^2 - 24t + 77$

(i) $\frac{dy}{dt} = 60t - 24 \quad \frac{dy}{dt} = 0 \Rightarrow t = \frac{24}{60} = \frac{2}{5}$

$\frac{d^2y}{dt^2} = 60 > 0$ so $t = \frac{2}{5}$ gives a minimum (2 marks)

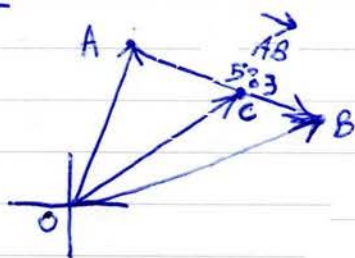
(b)(ii) when $t = \frac{2}{5}$,

$$\begin{aligned} y &= 30\left(\frac{2}{5}\right)^2 - 24\left(\frac{2}{5}\right) + 77 \\ &= 30\left(\frac{4}{25}\right) - 24\left(\frac{2}{5}\right) + 77 \\ &= \frac{24}{5} - \frac{48}{5} + 77 = \frac{361}{5} \quad (1 \text{ mark}) \end{aligned}$$

$$|\vec{AB}| = \sqrt{y} = \sqrt{\frac{361}{5}} = \frac{19\sqrt{5}}{5} \quad (1 \text{ mark})$$

(c)

$$\vec{OC} = \vec{OA} + \frac{5}{8}\vec{AB}$$



$$= \begin{pmatrix} 8 \\ -3 \\ 2 \end{pmatrix} + \frac{5}{8} \begin{pmatrix} \frac{2}{5} - 8 \\ 2\left(\frac{2}{5}\right) + 3 \\ 5\left(\frac{2}{5}\right) - 2 \end{pmatrix} \quad \text{Given } \vec{AB} \text{ is a minimum } (t = \frac{2}{5})$$

$$= \begin{pmatrix} 8 - \frac{19}{4} \\ -3 + \frac{19}{8} \\ 2 + 0 \end{pmatrix} = \begin{pmatrix} \frac{13}{4} \\ -\frac{5}{8} \\ 2 \end{pmatrix} \quad (2 \text{ marks})$$