

15. A curve has equation $y = f(x)$, where

$$f(x) = \frac{7xe^x}{\sqrt{e^{3x} - 2}} \quad x > \ln \sqrt[3]{2}$$

(a) Show that

$$f'(x) = \frac{7e^x(e^{3x}(2-x) + Ax + B)}{2(e^{3x} - 2)^{\frac{3}{2}}}$$

where A and B are constants to be found.

(5)

(a) Quotient Rule: $f(x) = \frac{u}{v}$

$$f'(x) = \frac{vu' - uv'}{v^2}$$

$$u = 7xe^x \quad u' = 7e^x + 7xe^x \quad (\text{by Product Rule})$$

$$v = (e^{3x} - 2)^{\frac{1}{2}} \quad v' = \frac{1}{2}(e^{3x} - 2)^{-\frac{1}{2}} \times e^{3x} \times 3 \quad (\text{by Chain Rule})$$

$$v^2 = e^{3x} - 2$$

$$\text{So, } f'(x) = \frac{(e^{3x} - 2)^{\frac{1}{2}}(7e^x + 7xe^x) - (7xe^x)\left(\frac{3}{2}e^{3x}(e^{3x} - 2)^{-\frac{1}{2}}\right)}{(e^{3x} - 2)}$$

$$= \frac{7e^x \left((e^{3x} - 2)^{\frac{1}{2}}(1+x) - x \left(\frac{3}{2} e^{3x} (e^{3x} - 2)^{-\frac{1}{2}} \right) \right)}{(e^{3x} - 2)}$$

$$\left(\times \frac{(e^{3x} - 2)^{\frac{1}{2}}}{(e^{3x} - 2)^{\frac{1}{2}}} \right) = \frac{7e^x \left((e^{3x} - 2)(1+x) - x \left(\frac{3}{2} e^{3x} \right) \right)}{(e^{3x} - 2)^{\frac{3}{2}}}$$

$$= \frac{7e^x (e^{3x} + xe^{3x} - 2 - 2x - \frac{3}{2}xe^{3x})}{(e^{3x} - 2)^{\frac{3}{2}}}$$

$$\left(\times \frac{2}{2} \right) = \frac{7e^x (2e^{3x} + 2xe^{3x} - 4 - 4x - 3xe^{3x})}{2(e^{3x} - 2)^{\frac{3}{2}}}$$

$$= \frac{7e^x (e^{3x}(2-x) - 4x - 4)}{2(e^{3x} - 2)^{\frac{3}{2}}}$$

(b) Hence show that the x coordinates of the turning points of the curve are solutions of the equation

$$x = \frac{2e^{3x} - 4}{e^{3x} + 4} \quad (2)$$

(b) $f'(x) = 0$ when numerator = 0

$$7e^x (e^{3x}(2-x) - 4x - 4) = 0$$

$$\Rightarrow e^{3x}(2-x) - 4x - 4 = 0, \text{ because } e^x > 0$$

$$2e^{3x} - xe^{3x} - 4x - 4 = 0$$

$$2e^{3x} - 4 = xe^{3x} + 4x$$

$$x(e^{3x} + 4) = 2e^{3x} - 4$$

$$x = \frac{2e^{3x} - 4}{e^{3x} + 4}$$

The equation $x = \frac{2e^{3x} - 4}{e^{3x} + 4}$ has two positive roots α and β where $\beta > \alpha$

A student uses the iteration formula

$$x_{n+1} = \frac{2e^{3x_n} - 4}{e^{3x_n} + 4}$$

in an attempt to find approximations for α and β

Diagram 1 shows a plot of part of the curve with equation $y = \frac{2e^{3x} - 4}{e^{3x} + 4}$ and part of the line with equation $y = x$

Using Diagram 1

(c) draw a staircase diagram to show that the iteration formula starting with $x_1 = 1$ can be used to find an approximation for β

(1)

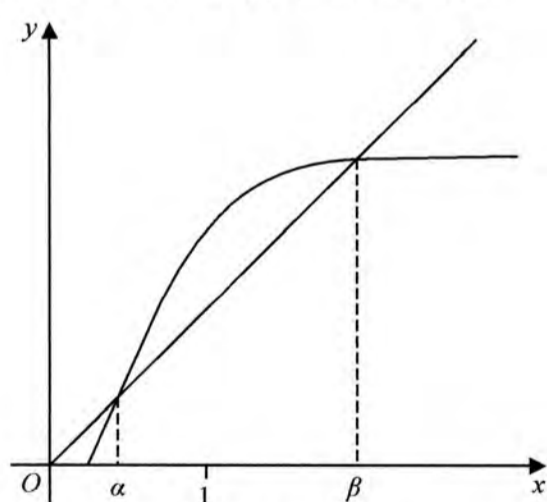


Diagram 1

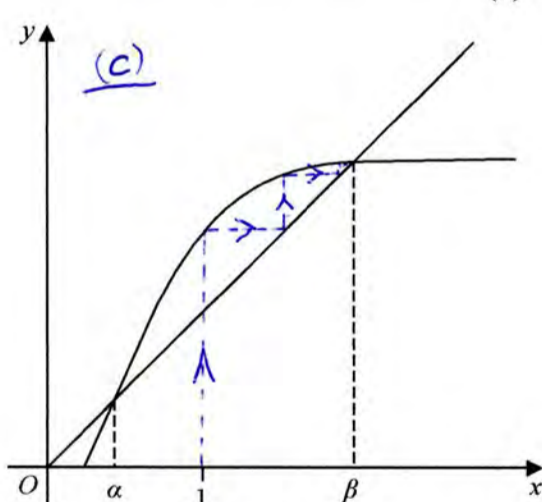


Diagram 1

Use the iteration formula with $x_1 = 1$, to find, to 3 decimal places,

(d) (i) the value of x_2

(ii) the value of β

(3)

$$(d) \quad x_1 = 1.0000$$

$$\left(x_{n+1} = \frac{2e^{(3 \times \text{Ans})} - 4}{e^{(3 \times \text{Ans})} + 4} \right)$$

$$(i) \quad x_2 = 1.5017... = 1.502 \text{ (3dp)}$$

$$(ii) \quad x_3 = 1.8730...$$

$$x_4 = 1.9570...$$

$$x_5 = 1.9665...$$

$$x_6 = 1.9674...$$

$$x_7 = 1.96756...$$

$$x_8 = 1.96756...$$

x_7 & x_8 agree to 3dp
so $\beta = 1.968$ (3dp)

Using a suitable interval and a suitable function that should be stated

(e) show that $\alpha = 0.432$ to 3 decimal places.

(2)

$$(e) \quad g(x) = \frac{2e^{3x} - 4}{e^{3x} + 4} - x$$

$$g(0.4315) = -0.000297...$$

$$g(0.4325) = +0.000947...$$

change of sign and $g(x)$ is continuous
so $\alpha = 0.432$ (3dp)