Question	Scheme	Marks	AOs
13(a)	$x = a\sin^2\theta \Rightarrow \dfrac{dx}{d\theta} = 2a\sin\theta\cos\theta$	B1	1.1b
	$\int x^{\frac{1}{2}}\sqrt{a-x}\,dx = \int \sqrt{a}\sin\theta\sqrt{a-a\sin^2\theta}\times 2a\sin\theta\cos\theta\{d\theta\}$	M1	2.1
	$= \int \sqrt{a}\sin\theta\sqrt{a}\cos\theta\times 2a\sin\theta\cos\theta\,d\theta = 2a^2\int \sin^2\theta\cos^2\theta\,d\theta$ $= 2a^2\int\left(\frac{1}{2}\sin 2\theta\right)^2 d\theta$	dM1	3.1a
	Replaces or considers limits $\{x=0\Rightarrow\}\ \theta=0,\ \{x=a\Rightarrow\}\ \theta=\frac{\pi}{2}$ $= \frac{1}{2}a^2\int_0^{\frac{\pi}{2}}\sin^2 2\theta\,d\theta$ *	A1*	1.1b
		(4)	
(b)	$\ldots\int \sin^2 2\theta\,d\theta \rightarrow \ldots\int \frac{1-\cos 4\theta}{2}\,d\theta$	M1	1.1b
	$\rightarrow \ldots\left(\frac{\theta}{2} - \frac{1}{8}\sin 4\theta\right)$	dM1	2.1
	$\mu\int \sin^2 2\theta\,d\theta \rightarrow \frac{\mu}{2}\left(\theta - \frac{1}{4}\sin 4\theta\right)$	A1	1.1b
	$\left\{\int_0^a x^{\frac{1}{2}}\sqrt{a-x}\,dx\right\} = \frac{a^2}{4}\left(\frac{\pi}{2}-0-0\right) = \frac{1}{8}\pi a^2$	A1	1.1b
		(4)	
			(8 marks)

Notes

(a)

B1: $\dfrac{dx}{d\theta} = 2a\sin\theta\cos\theta$ or $dx = 2a\sin\theta\cos\theta\,d\theta$ o.e. seen or implied by their substitution.

Note that writing $x = a\sin^2\theta = \dfrac{a}{2}(1-\cos 2\theta) \rightarrow \dfrac{dx}{d\theta} = a\sin 2\theta$ is correct. Condone use of $\dfrac{dx}{da}$

M1: Attempts to substitute, fully replacing $x^{\frac{1}{2}}$ and $\sqrt{a-x}$ with θ's and dx with their dx = …

Look for $x^{\frac{1}{2}}\sqrt{a-x}\,dx \rightarrow f(\theta)g(\theta)h(\theta)$ where

- $f(\theta)$ is an attempt at $\sqrt{a\sin^2\theta}$ e.g. allow $a\sin\theta$ but just $a\sin^2\theta$ is not condoned
- $g(\theta)$ is an attempt at $\sqrt{a-a\sin^2\theta}$ but not $\sqrt{a}-\sqrt{a\sin^2\theta}$ unless $\sqrt{a-a\sin^2\theta}$ is attempted first
- $h(\theta)$ = their dx or their $\dfrac{dx}{d\theta}$ or $\dfrac{1}{\text{their } \frac{dx}{d\theta}}$ (in terms of θ only but condone da seen)

Condone slips provided the intention is clear, e.g. $x^{\frac{1}{2}} \rightarrow \sqrt{a}\sin^2\theta$ but x must be eliminated.
There is no need to replace the limits and condone the absence of $d\theta$ and/or the integral sign for this mark.

dM1: Attempts to use $\sin 2\theta = 2\sin\theta\cos\theta$ to convert an integral of the form $\int \sin^2\theta\cos^2\theta\,\mathrm{d}\theta$ or e.g. the form $\int \sin\theta\cos\theta\sin 2\theta\,\mathrm{d}\theta$ to $\int \ldots\sin^2 2\theta\,\mathrm{d}\theta$

There is no need to replace the limits and condone the absence of $\mathrm{d}\theta$ and/or the integral sign for this mark.

A1*: Replaces or considers limits $\{x = 0 \Rightarrow\}\,\theta = 0,\ \{x = a \Rightarrow\}\,\theta = \frac{\pi}{2}$ at some stage before the given answer and proceeds with no errors to the given answer. The replaced limits may appear with their integral symbol and do not have to be justified and do not have to appear on every line. Condone infrequent slips in notation, e.g. $\sin\theta^2$ in a line as long as it is not consistently poor. You must see the integral sign with the correct limits and the $\mathrm{d}\theta$ together in the given answer.

(b)

M1: Adopts an appropriate strategy by using the double angle identity to obtain an integrable form $\ldots\int \sin^2 2\theta\,\mathrm{d}\theta \to \ldots\int \frac{\pm1\pm\cos 4\theta}{2}\,\mathrm{d}\theta$ which may be seen as $\lambda\int \sin^2 2\theta\,\mathrm{d}\theta \to \frac{\lambda}{2}\int \pm1\pm\cos 4\theta\,\mathrm{d}\theta$ with the $\frac{1}{2}$ absorbed into their coefficient of the integral.

dM1: Integrates into the form $\pm p\theta \pm q\sin 4\theta$

A1: Correct integration of $\mu\int \sin^2 2\theta\,\mathrm{d}\theta$ to $\frac{\mu}{2}\left(\theta - \frac{1}{4}\sin 4\theta\right)$. Here μ may be 1.

Condone lack of limits here.

A1: Applies limits to the correct integral and proceeds to $\frac{1}{8}\pi a^2$ following correct work.

There is no need to see 0 substituted in and condone any omission of integral signs and/or $\mathrm{d}\theta$

Note that $\frac{a^2}{4}\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{2}} = \frac{1}{8}\pi a^2$ is incorrect and scores M1dM1A0A0

Use of $\sin^2 2\theta = \frac{\pm1\pm\cos k\theta}{2}$ with $k \neq 4$ scores M0dM0A0A0 but may lead to $\frac{1}{8}\pi a^2$

Condone use of x in place of θ e.g. $\frac{a^2}{4}\left[x - \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{2}} = \frac{1}{8}\pi a^2$

See overleaf for some alternative approaches.

Alternative 13(a) working backwards:

$$\frac{1}{2}a^2\int_0^{\frac{\pi}{2}}\sin^2 2\theta\,\mathrm{d}\theta \qquad\qquad \frac{\mathrm{d}x}{\mathrm{d}\theta}=2a\sin\theta\cos\theta \text{ score B1 (as in main scheme)}$$

$$=\frac{1}{2}a^2\int_0^{\frac{\pi}{2}}(2\sin\theta\cos\theta)(2\sin\theta\cos\theta)\,\mathrm{d}\theta$$

$$=a\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\,\mathrm{d}x$$

Score M1 here for using the double angle identity and replacing $\ldots\sin\theta\cos\theta\ \mathrm{d}\theta$ with $\mathrm{d}x$

$$=a\int_0^a\sqrt{\frac{x}{a}}\sqrt{1-\frac{x}{a}}\,\mathrm{d}x$$

Score dM1 here for a full attempt to replace all trig leading to everything in terms of x only

Must come from the form $\int \ldots\sin\theta\cos\theta\,\mathrm{d}x$

A1 fully correct with limits replaced / considered before the final line and the final line fully correct with limits, integral sign and dx as per the main scheme.

Alternative 13(b) via IBP Way 1:

$$\int_0^{\frac{\pi}{2}}\sin^2 2\theta\,\mathrm{d}\theta \qquad\qquad \left\{\begin{matrix} u=\sin^2 2\theta & v'=1 \\ u'=4\sin 2\theta\cos 2\theta & v=\theta\end{matrix}\right\}$$

$$=\left[\theta\sin^2 2\theta\right]_0^{\frac{\pi}{2}}-4\int_0^{\frac{\pi}{2}}\theta\sin 2\theta\cos 2\theta\,\mathrm{d}\theta$$

$$=\left[\theta\sin^2 2\theta\right]_0^{\frac{\pi}{2}}-2\int_0^{\frac{\pi}{2}}\theta\sin 4\theta\,\mathrm{d}\theta \qquad\qquad \left\{\begin{matrix} u=\theta & v'=\sin 4\theta \\ u'=1 & v=-\dfrac{\cos 4\theta}{4}\end{matrix}\right\}$$

$$=\left[\theta\sin^2 2\theta\right]_0^{\frac{\pi}{2}}-2\left(\left[-\frac{\theta\cos 4\theta}{4}\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}-\frac{\cos 4\theta}{4}\mathrm{d}\theta\right) \qquad \text{Score M1 here.}$$

$$=\left[\theta\sin^2 2\theta\right]_0^{\frac{\pi}{2}}-2\left[-\frac{\theta\cos 4\theta}{4}+\frac{\sin 4\theta}{16}\right]_0^{\frac{\pi}{2}} \quad \text{Score dM1 here, A1 if correct (ignoring limits).}$$

$$=\left[\theta\sin^2 2\theta + \frac{\theta\cos 4\theta}{2} - \frac{\sin 4\theta}{8}\right]_0^{\frac{\pi}{2}}$$

$$=\left(0+\frac{\pi}{4}-0\right)-(0+0-0)=\frac{\pi}{4} \ \rightarrow \left\{\int_0^a x^{\frac{1}{2}}\sqrt{a-x}\ \mathrm{d}x=\right\}=\frac{1}{8}\pi a^2 \text{ score A1* (with no errors).}$$

Alternative 13(b) via IBP Way 2:

$$\int_0^{\frac{\pi}{2}} \sin^2 2\theta \,\mathrm{d}\theta \qquad \left\{\begin{matrix} u = \sin 2\theta & v' = \sin 2\theta \\ u' = 2\cos 2\theta & v = -\dfrac{\cos 2\theta}{2} \end{matrix}\right\}$$

$$= \left[-\frac{\sin 2\theta \cos 2\theta}{2}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\cos^2 2\theta \,\mathrm{d}\theta$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin^2 2\theta \,\mathrm{d}\theta + \int_0^{\frac{\pi}{2}} \sin^2 2\theta \,\mathrm{d}\theta = \left[-\frac{\sin 2\theta \cos 2\theta}{2}\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos^2 2\theta \,\mathrm{d}\theta + \int_0^{\frac{\pi}{2}} \sin^2 2\theta \,\mathrm{d}\theta$$

$$\Rightarrow 2\int_0^{\frac{\pi}{2}} \sin^2 2\theta \,\mathrm{d}\theta = \left[-\frac{\sin 2\theta \cos 2\theta}{2}\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} 1 \,\mathrm{d}\theta$$ Score M1 here.

$$\Rightarrow 2\int_0^{\frac{\pi}{2}} \sin^2 2\theta \,\mathrm{d}\theta = \left[-\frac{\sin 2\theta \cos 2\theta}{2} + \theta\right]_0^{\frac{\pi}{2}}$$ Score dM1 here, A1 if correct including the 2 (ignoring limits).

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin^2 2\theta \,\mathrm{d}\theta = \frac{1}{2}\left(0 + \frac{\pi}{2}\right) - \frac{1}{2}(0+0) = \frac{\pi}{4}$$

$$\rightarrow \left\{\int_0^a x^{\frac{1}{2}}\sqrt{a-x}\,\mathrm{d}x = \right\} = \frac{1}{8}\pi a^2$$ score A1* (with no errors).