

14. (a) Use the substitution  $u = 1 + \sin^2 x$  to show that

$$\int_0^{\frac{\pi}{6}} \frac{8 \tan x}{1 + \sin^2 x} dx = \int_p^q \frac{4}{u(2-u)} du$$

where  $p$  and  $q$  are constants to be found.

(5)

(b) Hence, using algebraic integration, show that

$$\int_0^{\frac{\pi}{6}} \frac{8 \tan x}{1 + \sin^2 x} dx = \ln A$$

where  $A$  is a rational number to be found.

(6)

(a)  $u = 1 + \sin^2 x$

$$\frac{du}{dx} = 2 \sin x \cos x \text{ by Chain Rule (1 mark)}$$

$$\int_{x=0}^{x=\frac{\pi}{6}} \frac{8 \tan x}{1 + \sin^2 x} dx = \int_{x=0}^{x=\frac{\pi}{6}} \frac{8 \tan x}{u} \frac{dx}{du} du$$

$$= \int_{x=0}^{x=\frac{\pi}{6}} \frac{8 \sin x}{\cos x} \times \frac{1}{u} \times \frac{1}{2 \sin x \cos x} du \quad (1 \text{ mark})$$

$$= \int_{x=0}^{x=\frac{\pi}{6}} \frac{4}{\cos^2 x u} du = \int_{x=0}^{x=\frac{\pi}{6}} \frac{4}{(1 - \sin^2 x) u} du$$

$$= \int_{x=0}^{x=\frac{\pi}{6}} \frac{4}{(2 - 1 - \sin^2 x) u} du = \int_{x=0}^{x=\frac{\pi}{6}} \frac{4}{(2 - u) u} du \quad (2 \text{ marks})$$

so,  $\int_1^{\frac{5}{4}} \frac{4}{u(2-u)} du$  change limits  $\begin{cases} x & u = 1 + \sin^2 x \\ 0 & 1 \quad (p) \\ \frac{\pi}{6} & \frac{5}{4} \quad (q) \end{cases}$  (1 mark)

(b) Partial Fractions:  $\frac{4}{u(2-u)} \equiv \frac{A}{u} + \frac{B}{2-u}$

$$4 \equiv A(2-u) + Bu \quad \text{when } u=2, 4=2B \Rightarrow B=2 \quad \text{when } u=0, 4=2A \Rightarrow A=2$$

so,  $\int_1^{\frac{5}{4}} \left( \frac{2}{u} + \frac{2}{2-u} \right) du$  (2 marks)  $= [2 \ln u - 2 \ln(2-u)]_1^{\frac{5}{4}}$  (2 marks)

$$= 2 \ln\left(\frac{5}{4}\right) - 2 \ln\left(\frac{3}{4}\right) - 2 \ln 1 - 2 \ln 1 = 2 \ln\left(\frac{5}{4}\right) - 2 \ln\left(\frac{3}{4}\right) - 0 - 0$$
 (1 mark)

$$= 2 \ln\left(\frac{5/4}{3/4}\right) = 2 \ln\left(\frac{5}{3}\right) = \ln\left(\frac{5}{3}\right)^2$$

$$= \ln\left(\frac{25}{9}\right) \quad (1 \text{ mark})$$