Summary of key points

4 You can prove a mathematical statement is true by **deduction**. This means starting from known facts or definitions, then using logical steps to reach the desired conclusion.

5 In a mathematical proof you must
- State any information or assumptions you are using
- Show every step of your proof clearly
- Make sure that every step follows logically from the previous step
- Make sure you have covered all possible cases
- Write a statement of proof at the end of your working

6 To prove an identity you should
- Start with the expression on one side of the identity
- Manipulate that expression algebraically until it matches the other side
- Show every step of your algebraic working

7 You can prove a mathematical statement is true by **exhaustion**. This means breaking the statement into smaller cases and proving each case separately.

8 You can prove a mathematical statement is not true by a **counter-example**. A counter-example is one example that does not work for the statement. You do not need to give more than one example, as one is sufficient to disprove a statement.

Proof

If you need to **prove** a statement in your exam, you need to construct a **logical argument** to show that it is always true. You need to know these two methods of proof.

1 Proof by deduction

Also called **direct proof**. You use known facts and follow logical steps to reach a conclusion.

2 Proof by exhaustion

You consider **each of the possible cases** separately, in order to show that something is true in every case.

Worked example

x and y are rational numbers.

Prove that the mean of x and y is also a rational number. **(3 marks)**

Let $x = \frac{a}{b}$ and $y = \frac{c}{d}$, where a, b, c and d are integers. The mean of x and y is:

$$\frac{x+y}{2} = \frac{1}{2}\left(\frac{a}{b} + \frac{c}{d}\right) = \frac{1}{2}\left(\frac{ad}{bd} + \frac{bc}{bd}\right) = \frac{ad+bc}{2bd}$$

Since a, b, c and d are integers, $(ad + bc)$ and $2bd$ must also be integers, so $\frac{x+y}{2}$ is a rational number.

Proof checklist

- ✓ Write down any rules, information or assumptions you need to use.
- ✓ Show each step of your working clearly.
- ✓ Make sure your steps follow logically from each other.
- ✓ Write down what you have proved at the end of your working.

Rational numbers are numbers that can be written in the form $\frac{p}{q}$, where p and q are integers.

Problem solved!

You can prove this result by considering the cases where x and y are positive or negative separately. You know you have covered **all possible cases** because at least one of the following statements must be true:

- x is negative
- y is negative
- both x and y are non-negative.

Remember to write down what you have proved at the end of your working.

You will need to use problem-solving skills throughout your exam – **be prepared!**

Worked example

Prove that $(xy + 1)^2 + (x - 1)^2 + (y - 1)^2 \geqslant 1$ for all real values of x and y. **(4 marks)**

Each of $(xy + 1)^2$, $(x - 1)^2$ and $(y - 1)^2$ is $\geqslant 0$ for all real values of x and y.

If $x < 0$ then $(x - 1)^2 > 1$

If $y < 0$ then $(y - 1)^2 > 1$

If x and y are both $\geqslant 0$ then $(xy + 1)^2 \geqslant 1$

All three terms are $\geqslant 0$ and at least one must be $\geqslant 1$, so the whole expression must be $\geqslant 1$, as required.

Now try this

1 $f(x) = 2x^3 + 5x^2 - 4$

A student is attempting to use the factor theorem to prove that $(x + 2)$ is a factor of $f(x)$. The student writes the following working:

$$\begin{aligned} f(-2) &= 2(-2)^3 + 5(-2)^2 - 4 \\ &= -16 + 20 - 4 \\ &= 0 \end{aligned}$$

Answer

(a) Explain why this proof is incomplete. **(1 mark)**

(b) Write an additional line of working to complete the proof. Answer **(1 mark)**

2 (a) Prove by exhaustion that $2n^2 + 11$ is a prime number for all $n \in \mathbb{N}$, $n \leqslant 10$. Answer **(2 marks)**

(b) By means of a counterexample disprove the statement:

$n^2 + n + 17$ is a prime number for all $n \in \mathbb{N}$. Answer **(2 marks)**

3 Prove that $(4 - x)^2 \geqslant 7 - 2x$ for all real values of x. Answer **(3 marks)**