Summary of key points

1 Pascal's triangle is formed by adding adjacent pairs of numbers to find the numbers on the next row.

2 The $(n + 1)$th row of Pascal's triangle gives the coefficients in the expansion of $(a + b)^n$.

3 $n! = n \times (n - 1) \times (n - 2) \times \ldots \times 3 \times 2 \times 1$.

4 You can use factorial notation and your calculator to find entries in Pascal's triangle quickly.

- The number of ways of choosing r items from a group of n items is written as ${}^nC_r$ or $\binom{n}{r}$: ${}^nC_r = \binom{n}{r} = \dfrac{n!}{r!(n-r)!}$
- The rth entry in the nth row of Pascal's triangle is given by ${}^{n-1}C_{r-1} = \binom{n-1}{r-1}$.

5 The binomial expansion is:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n \quad (n \in \mathbb{N})$$

where $\binom{n}{r} = {}^nC_r = \dfrac{n!}{r!(n-r)!}$

6 In the expansion of $(a + b)^n$ the general term is given by $\binom{n}{r}a^{n-r}b^r$.

7 If x is small, the first few terms in the binomial expansion can be used to find an approximate value for a complicated expression.

The binomial expansion

The binomial expansion is a formula that lets you **expand brackets** easily. This is how the binomial expansion will appear in the formulae booklet in your AS exam.

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n \quad (n \in \mathbb{N})$$

where $\binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$

This means n **factorial**. $n! = n \times (n-1) \times \ldots \times 3 \times 2 \times 1$

The expansion is valid as long as n is a **positive integer**.

$\binom{n}{r}$ or ${}^nC_r$ means n CHOOSE r – use the nCr function on your calculator to work it out.

Worked example

Find the first 3 terms, in ascending powers of x, of the binomial expansion of $(2-3x)^6$. Give each term in its simplest form. **(4 marks)**

$a = 2,\ b = -3x$

$(a+b)^6 = a^6 + \binom{6}{1}a^5b + \binom{6}{2}a^4b^2 + \ldots$

$(2-3x)^6 = 2^6 + \binom{6}{1} \times 2^5 \times (-3x) + \binom{6}{2} \times 2^4 \times (-3x)^2 + \ldots$

$= 64 - 576x + 2160x^2 + \ldots$

Be careful if either a or b is negative. Always use brackets if you are substituting anything more complicated than a positive whole number.

Problem solved!

$b = px$, so in the third term you need to square **all** of px.
$(px)^2 = p^2x^2$.

To work out $\binom{10}{2}$ or ${}^{10}C_2$ on your calculator, type 10 nCr 2 =.

Make sure you **simplify** each term as much as possible. Don't leave any powers of numbers or multiplication signs in your final answer.

You will need to use problem-solving skills throughout your exam – **be prepared!**

Worked example

(a) Find the first 3 terms, in ascending powers of x, of the binomial expansion of $(1+px)^{10}$, where p is a non-zero constant. Give each term in its simplest form. **(2 marks)**

$a = 1,\ b = px$

$(a+b)^{10} = a^{10} + \binom{10}{1}a^9b + \binom{10}{2}a^8b^2 + \ldots$

$(1+px)^{10} = 1^{10} + \binom{10}{1} \times 1^9 \times px + \binom{10}{2} \times 1^8 \times (px)^2 + \ldots$

$= 1 + 10px + 45p^2x^2 + \ldots$

(b) Given that the coefficient of x^2 is 9 times the coefficient of x, find the value of p. **(2 marks)**

$45p^2 = 9(10p)$ so $45p^2 - 90p = 0$

$45p(p-2) = 0$

~~$p = 0$~~ $p = 2$

Now try this

1 Find the first 4 terms, in ascending powers of x, of each of these binomial expansions, giving each term in its simplest form.

(a) $(1+3x)^9$ Answer **(3 marks)**

(b) $(2+5x)^4$ Answer **(4 marks)**

(c) $(3-x)^{12}$ Answer **(4 marks)**

2 (a) Find the first 3 terms, in ascending powers of x, of the binomial expansion of $(2+kx)^5$, where k is a constant. **(4 marks)** Answer

(b) Given that the coefficient of x is 48, find the value of k. **(2 marks)** Answer

(c) Write down the coefficient of x^2 **(1 mark)** Answer

Had a look ☐ Nearly there ☐ Nailed it! ☐

Solving binomial problems

You can use the binomial expansion to make **approximations** or to solve harder problems.

Worked example

Find the coefficient of x^5 in the binomial expansion of $\left(6 - \frac{x}{3}\right)^9$ **(2 marks)**

$a = 6$, $b = -\frac{x}{3}$, $n = 9$ so x^r term $= \binom{n}{r}a^{n-r}b^r$

x^5 term $= \binom{9}{5} \times 6^4 \times \left(-\frac{x}{3}\right)^5$

$= 126 \times 1296 \times \left(-\frac{1}{243}\right)x^5$

$= -672x^5$

The coefficient of x^5 is -672.

You can use this part of the binomial expansion of $(a + b)^n$ in the formulae booklet to find the x^r term without finding every term up to it:

$$\ldots + \binom{n}{r}a^{n-r}b^r + \ldots$$

The x part of each term comes from $b = -\frac{x}{3}$, so use $r = 5$ to get the x^5 term. Be careful with the fraction part: $\left(-\frac{x}{3}\right)^5 = \left(-\frac{1}{3}\right)^5 x^5$

Binomial approximations

You can use a binomial expansion to **estimate** values. This is especially useful if x is **small**. If x is less than 1, then **larger power** of x get **smaller**. By ignoring large powers of x you can find a simple approximation. For example:

$(1 + x)^{100} \approx 1 + 100x + 4950x^2$

This means 'is approximately equal to'.

Worked example

The first 4 terms of the binomial expansion of $\left(1 + \frac{x}{2}\right)^6$ are given below.

$\left(1 + \frac{x}{2}\right)^6 = 1 + 3x + \frac{15}{4}x^2 + \frac{5}{2}x^3 + \ldots$

Use the expansion to estimate the value of $(1.05)^6$ **(3 marks)**

If $x = 0.1$, then $\frac{x}{2} = 0.05$ and $\left(1 + \frac{x}{2}\right)^6 = (1.05)^6$

$(1.05)^6 \approx 1 + 3 \times (0.1) + \frac{15}{4} \times (0.1)^2 + \frac{5}{2} \times (0.1)^3 + \ldots$

$= 1 + 0.3 + 0.0375 + 0.0025 + \ldots$

$= 1.34$

Write down the value of x you need to substitute. You can check your answer with a calculator. $1.05^6 = 1.340095\ldots$ ✓

Worked example

The first 4 terms of the binomial expansion of $(2 - x)^7$ are given below.

$(2 - x)^7 = 128 - 448x + 672x^2 + 560x^3 + \ldots$

If x is small, so that x^2 and higher powers can be ignored, show that $(1 + x)(2 - x)^7 \approx 128 - 320x$ **(2 marks)**

$(1 + x)(128 - 448x + \ldots)$

$= 128 + 128x - 448x - ~~448x^2~~ + \ldots$

$= 128 - 320x$ ignoring x^2 and higher powers

You are going to **ignore** x^2 and higher powers, so you only need to consider the first two terms of the expansion of $(2 - x)^7$. All the other terms would give you x^2 or higher terms when the brackets are multiplied out.

Now try this

1 In the binomial expansion of $(1 + 2x)^{30}$, the coefficients of x^3 and x^4 are p and q respectively.

(a) Show that $p = 32480$ **(1 mark)** Answer

(b) Find the value of $\left(\frac{q}{p}\right)$ **(2 marks)** Answer

2 (a) Find the first 4 terms, in ascending powers of x, of the binomial expansion of $\left(1 + \frac{x}{4}\right)^{12}$ **(4 marks)** Answer

(b) Use your expansion to estimate the value of $(1.025)^{12}$, giving your answer to 4 decimal places. **(3 marks)** Answer