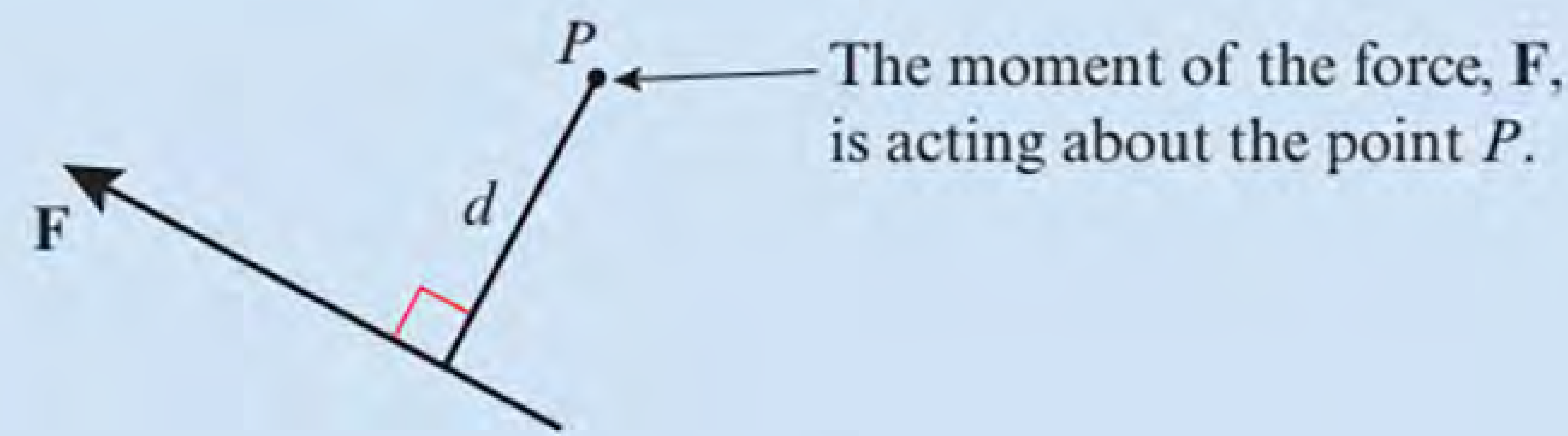
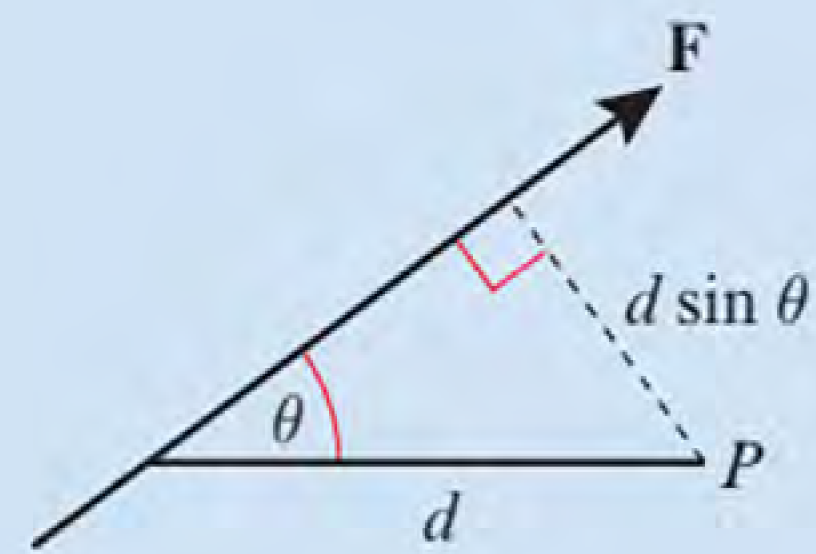
Summary of key points

1 Moment of **F** about $P = |\mathbf{F}| \times d$ clockwise



2 Moment of **F** about $P = |\mathbf{F}| \times d\sin\theta$ clockwise



3 The sum of the moments acting on a body is called the resultant moment.

4 When a rigid body is in equilibrium the resultant force in any direction is 0 N and the resultant moment about any point is 0 N m.

5 When a rigid body is on the point of tilting about a pivot, the reaction at any other support (or the tension in any other wire or string) is zero.

Summary of key points

1 A particle or rigid body is in static equilibrium if it is at rest and the resultant force acting on the particle is zero.

4 For a rigid body in static equilibrium:

- the body is stationary
- the resultant force in any direction is zero
- the resultant moment is zero.

Moments 1

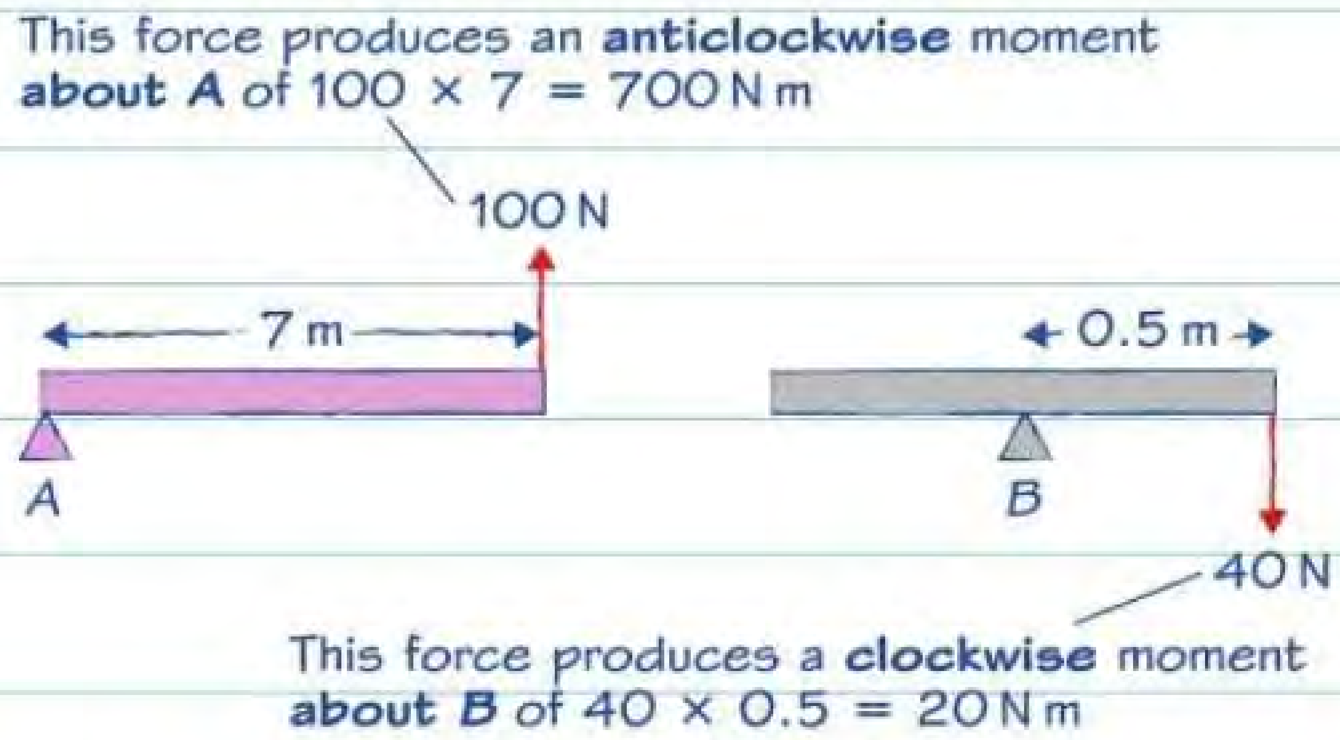
A moment is a measure of the **turning effect** of a force on a body. You can use moments to answer questions about **rods** in **equilibrium**.

Taking moments

If a force of magnitude F N acts at a perpendicular distance x m from a point P, the **moment of F about P** is Fx:

moment = force × distance

The **units** of a moment are newton-metres (N m). The turning effect of a moment can be either **anticlockwise** or **clockwise**.



Equilibrium

If a rod is **horizontal** and in equilibrium, then

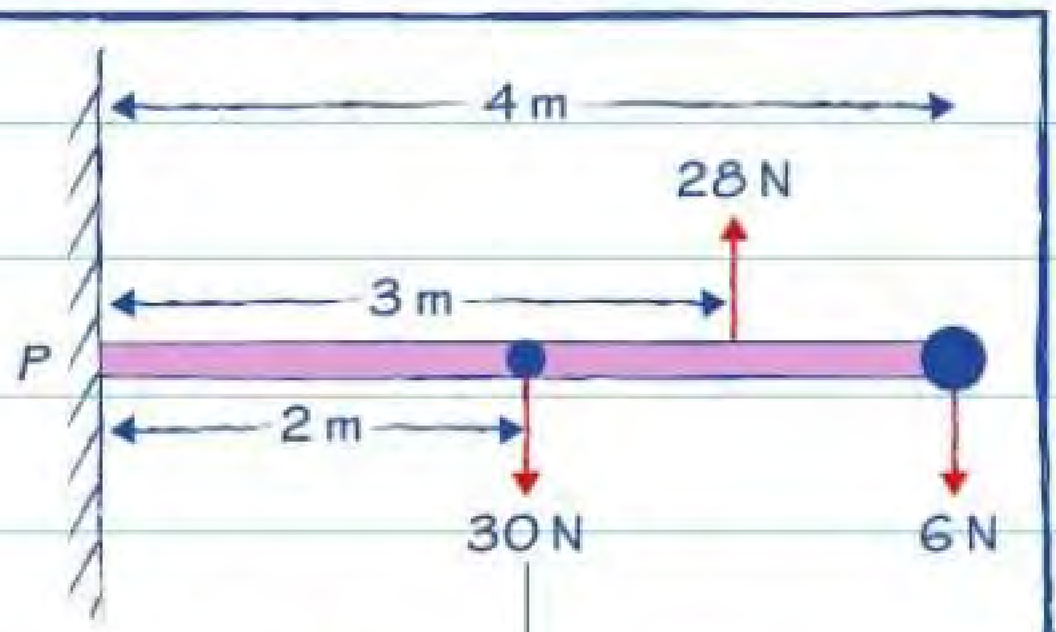
sum of clockwise moments = sum of anticlockwise moments

$$\sum \circlearrowleft = \sum \circlearrowright$$

The rod on the right has weight 30 N and is held in place by a vertical force of 28 N. A particle of weight 6 N is on the end of the rod.

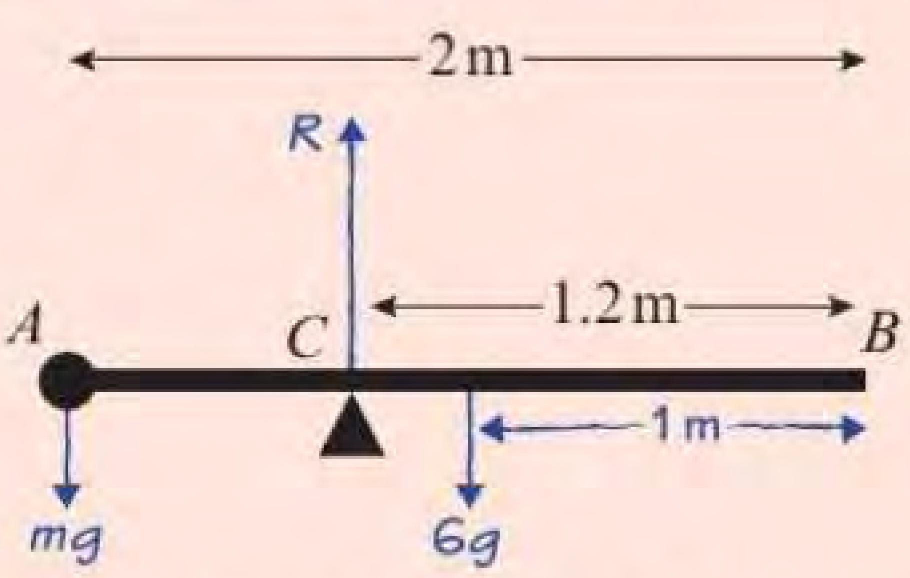
$\sum \circlearrowright$ moments about $P = 30 \times 2 + 6 \times 4$
$= 84$ N m

$\sum \circlearrowleft$ moments about $P = 28 \times 3 = 84$ N m



This rod is **uniform**. That means that its full weight acts at the **midpoint** of the rod. You can revise **non-uniform** rods on page 166.

Worked example



A uniform rod AB has length 2 m and mass 6 kg. A particle of mass m kg is attached to the rod at A. The rod is supported at C, where $CB = 1.2$ m, and the system is in equilibrium with AB horizontal. Find the value of m. **(4 marks)**

$\circlearrowleft$ moment about $C = 6g \times 0.2$

$\circlearrowright$ moment about $C = mg \times 0.8$

$\circlearrowleft$ moment $= \circlearrowright$ moment

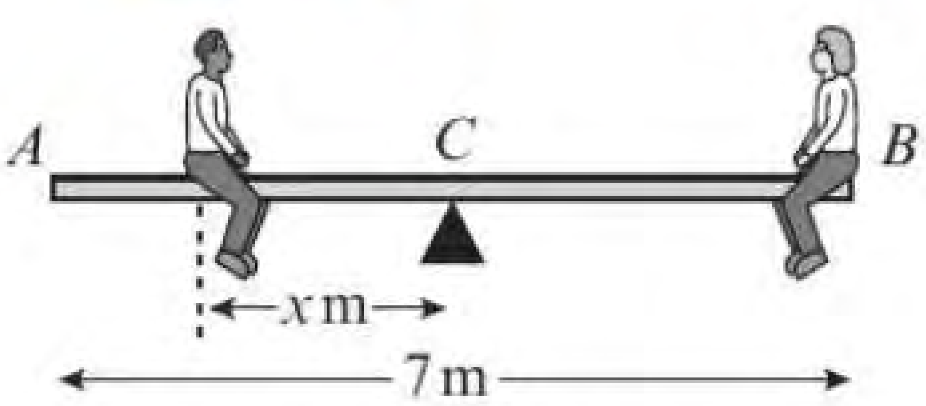
$1.2g = 0.8mg$

$m = 1.5$ kg

You can take moments about any point. Whichever point you choose, you can ignore any forces acting **at that point**. In this question there is a **normal reaction R** acting at C. The magnitude of this reaction will be $(mg + 6g)$ N, the total weight of the whole system. The easiest way to answer this question is to take moments about C. This means you can ignore the normal reaction R.

If you take moments about C you can ignore the weight of the beam.

Now try this



A seesaw consists of a beam AB of length 7 m supported by a smooth pivot at its centre C. Emma has mass 40 kg and sits on the end B. Paul has mass 50 kg and sits at a distance x m from C. The beam is modelled as a uniform rod.

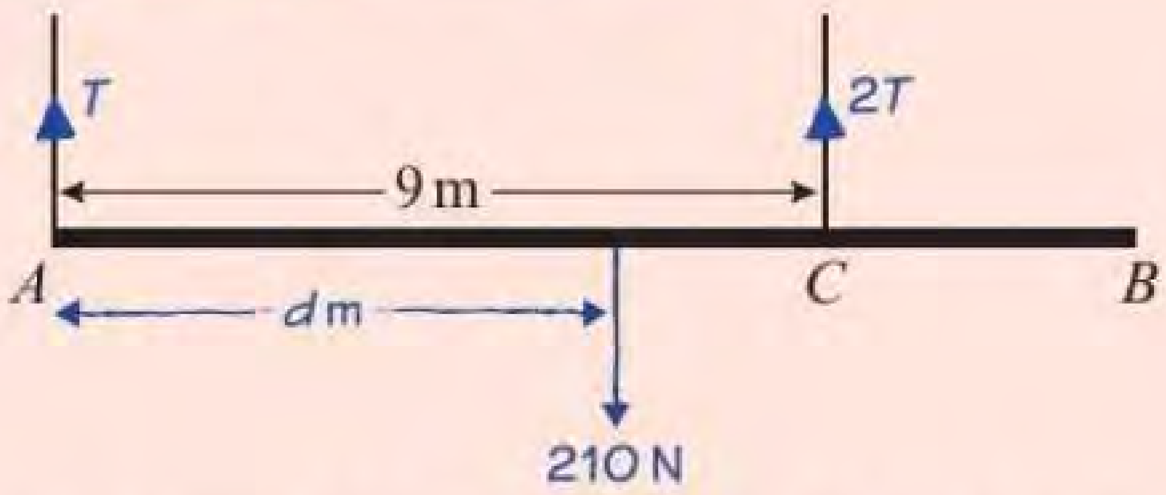
Find the value of x for which the seesaw rests in horizontal equilibrium. Answer **(3 marks)**

Had a look ☐ Nearly there ☐ Nailed it! ☐

Moments 2

If a rod is hanging in **equilibrium**, then the resultant force acting on it must be **zero**.
You sometimes need to **resolve vertically** to find unknown forces in moments questions.

Worked example



A steel girder AB has weight 210 N. It is held in equilibrium in a horizontal position by two vertical cables. One cable is attached to the end A. The other cable is attached to the point C on the girder, where $AC = 9$ m. The girder is modelled as a uniform rod, and the cables as light inextensible strings. Given that the tension in the cable at C is twice the tension in the cable at A

(a) find the tension in the cable at A **(2 marks)**

R(↑): $T + 2T - 210 = 0$

$T = 70$ N

(b) show that $AB = 12$ m. **(4 marks)**

Taking moments about A:

↺ $210 \times d = 2T \times 9$ ↻

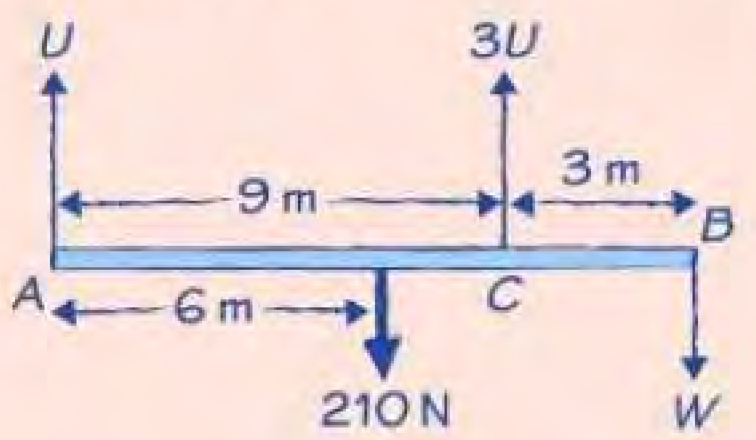
$210d = 1260$

$d = 6$

$AB = 2 \times d = 12$ m

A small load of weight W newtons is attached to the girder at B. The load is modelled as a particle. The girder remains in equilibrium in a horizontal position. The tension in the cable at C is now three times the tension in the cable at A.

(c) Find the value of W. **(7 marks)**



R(↑): $U + 3U - 210 - W = 0$

$4U - W = 210$ ①

Taking moments about B:

↺ $U \times 12 + 3U \times 3 = 210 \times 6$ ↻

$21U = 1260$

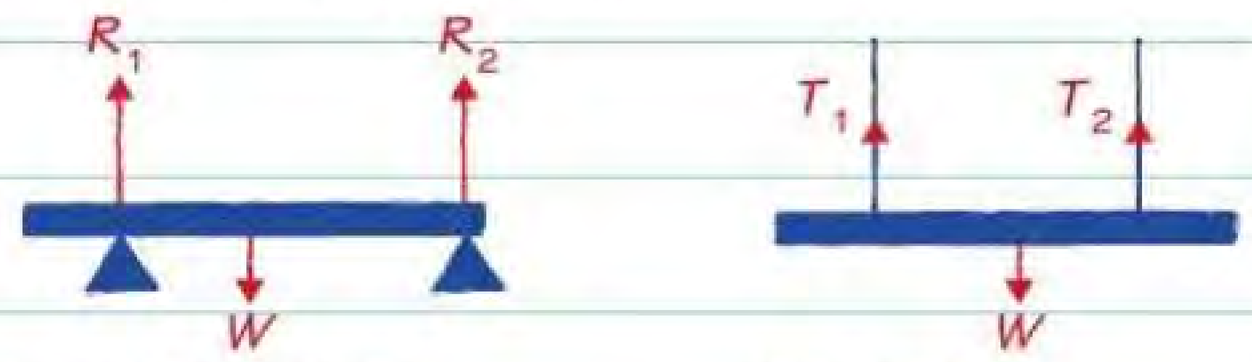
$U = 60$

Substituting into ①: $4 \times 60 - W = 210$

$W = 30$ N

Tensions and reactions

There are two ways a rod can be supported in a moments question:



This rod is resting on supports, which produce a **normal reaction** on the rod.

This rod is supported by ropes or cables, which have a **tension**.

The calculations work in **exactly the same way** in both situations.

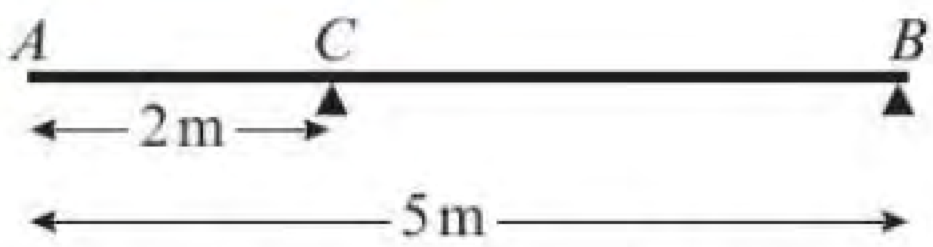
Start by labelling the diagram. The tension at C is twice the tension at A so label them T and $2T$.

(a) Resolve vertically for the **whole system** to work out the value of T.

(b) If you take moments about A you can ignore the tension in this cable. The rod is **uniform** so its weight acts at its midpoint. Find the distance from A to this point, then multiply by 2 to find the length of AB.

(c) This is a **new situation** so draw a new diagram showing the forces on the rod. The tensions will be different – choose a different letter from T to make sure you don't make a mistake.

Now try this



A uniform rod AB has mass 6 kg and length 5 m. The rod rests in horizontal equilibrium on two smooth supports. One support is at the end B. The other is at a point C on the beam, where $AC = 2$ m.

(a) Find the reaction on the beam at C.

Answer **(3 marks)**

A particle of mass 18 kg is attached to the rod at a point D. The beam remains in equilibrium. The reactions on the beam at C and B are now equal.

(b) Find the distance AD. **(7 marks)**

Answer

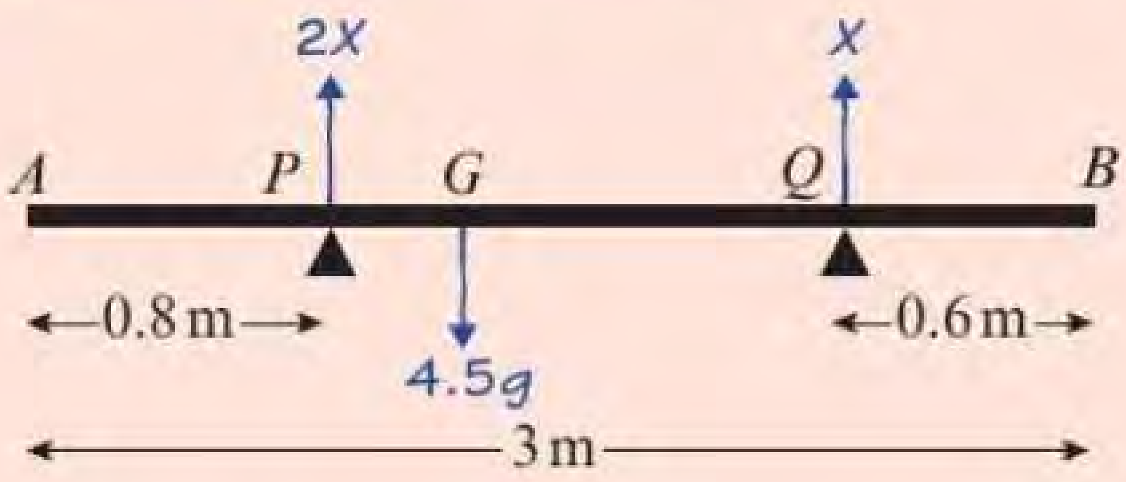
Centres of mass

The centre of mass of an object is the point at which you can assume the **weight** of the object acts. In a **uniform** rod, the centre of mass is the **midpoint** of the rod. If a rod is **non-uniform**, the weight can act at a different point.

Golden rule

If a rod is **non-uniform**, **do not assume** the weight acts at its midpoint.

Worked example



A non-uniform rod AB has length 3 m and mass 4.5 kg. The rod rests in equilibrium in a horizontal position, on two smooth supports at P and Q, where $AP = 0.8$ m and $QB = 0.6$ m. The centre of mass of the rod is at G. Given that the magnitude of the reaction of the support at P on the rod is twice the magnitude of the reaction of the support at Q on the rod, find

(a) the magnitude of the reaction of the support at Q on the rod **(3 marks)**

R(↑): $2X + X - 4.5g = 0$

$X = 1.5g = 15$ N (2 s.f.)

(b) the distance AG. **(4 marks)**

Taking moments about A:

$\circlearrowleft\ 4.5g \times AG = 2X \times 0.8 + X \times 2.4\ \circlearrowright$

$= 4X$

$4.5g \times AG = 6g$

$AG = \frac{4}{3}$ m

Problem solved!

Resolve vertically for the **whole system** to find X.

The rod is **non-uniform** so its centre of mass G is not necessarily the midpoint of the rod. You need to assume that the weight of the rod acts at G, and use moments to find the length AG.

You could take moments at any point to solve this problem. For example:

Taking moments about P:

$\circlearrowleft\ 4.5\text{g} \times PG = X \times 1.6\ \circlearrowright$

$= 2.4\text{g}$

$PG = \frac{8}{15}$ m

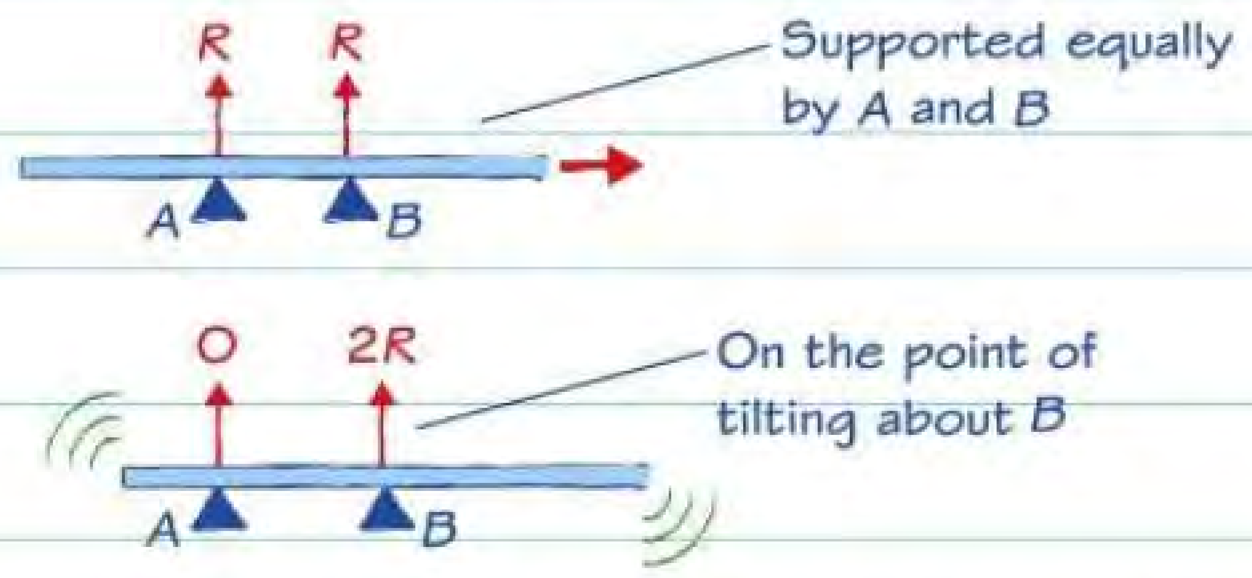
$AG = 0.8 + PG = \frac{4}{3}$ m

You will need to use problem-solving skills throughout your exam – **be prepared!**

On the point of tilting

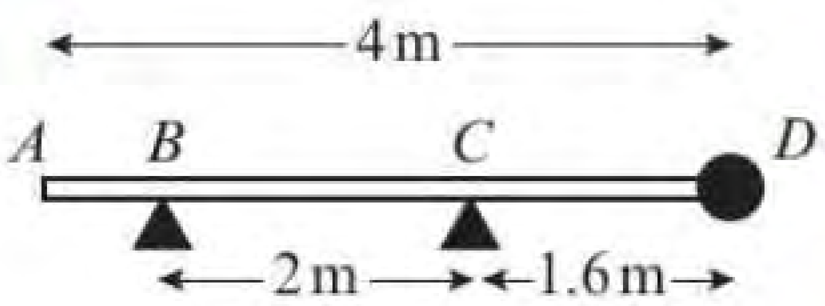
If a rod is on the point of tilting, one of the reactions on the rod will be equal to **zero**.

This rod is being slid to the right. Initially it rests on two supports A and B. Just before it tips over to the right, it will be resting **only** on B. The reaction at A will be zero.



Now try this

A plank AD weighs 20 kg and has length 4 m. The plank rests horizontally on two smooth supports at B and C, where $BC = 2$ m and $CD = 1.6$ m. A particle of mass m kg is attached to the plank at D. The plank is initally modelled as a uniform rod. With this model, the plank is on the point of tilting about C.



(a) The rod is on the point of tilting about C so the reaction at B will be zero.

(b) Assume that the weight of the plank acts at the centre of mass.

(a) Find the value of m. Answer **(6 marks)**

The plank is now modelled as a non-uniform rod. With the new model, the reaction at C is 75 N greater than the reaction at B.

(b) Find the distance of the centre of mass of the plank from D. Answer **(6 marks)**

Static rigid bodies

You might need to combine your knowledge of moments, forces and friction to solve problems involving rigid bodies in equilibrium. One of the most common questions of this type involves a ladder leaning against a wall.

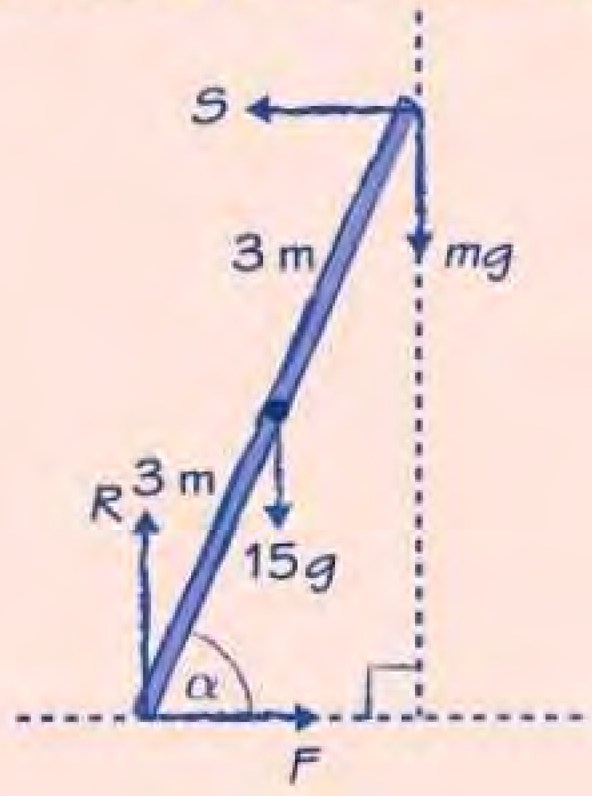
Worked example

A uniform ladder of mass 15 kg and length 6 m rests against a smooth vertical wall, with its lower end resting on rough horizontal ground. The coefficient of friction between the ground and the ladder is 0.4, and the ladder is inclined at an angle α to the horizontal, where $\tan\alpha = 2$.

A builder loads bricks through a window into a bucket attached to the top of the ladder. Find the maximum mass of bricks that can be loaded before the ladder starts to slip.

(8 marks)

Let the maximum mass of bricks be m kg.



$R(\rightarrow)$: $F = S$

$R(\uparrow)$: $R = 15g + mg$

Taking moments about the bottom of the ladder:

$15g \times 3\cos\alpha + mg \times 6\cos\alpha = S \times 6\sin\alpha$

$45g\left(\frac{1}{\sqrt{5}}\right) + 6mg\left(\frac{1}{\sqrt{5}}\right) = 6S\left(\frac{2}{\sqrt{5}}\right)$

$45g + 6mg = 12S$ (1)

When the ladder is on the point of slipping (in limiting equilibrium), $F = \mu R$.

Since $S = F = \mu R = 0.4R$, and $R = 15g + mg$, equation (1) becomes:

$45g + 6mg = 12 \times 0.4(15g + mg)$

$45 + 6m = 72 + 4.8m$

$m = 22.5$ kg

Problem solved!

It is essential to draw a diagram for problems such as this. Make sure you include:

- the normal reactions at the wall and the ground
  These are different so use **different** letters.
- the frictional force for any rough surface
  Frictional forces act so as to **oppose** the direction of motion.
- the weight of the ladder.
  If the ladder is uniform this will act at its midpoint.

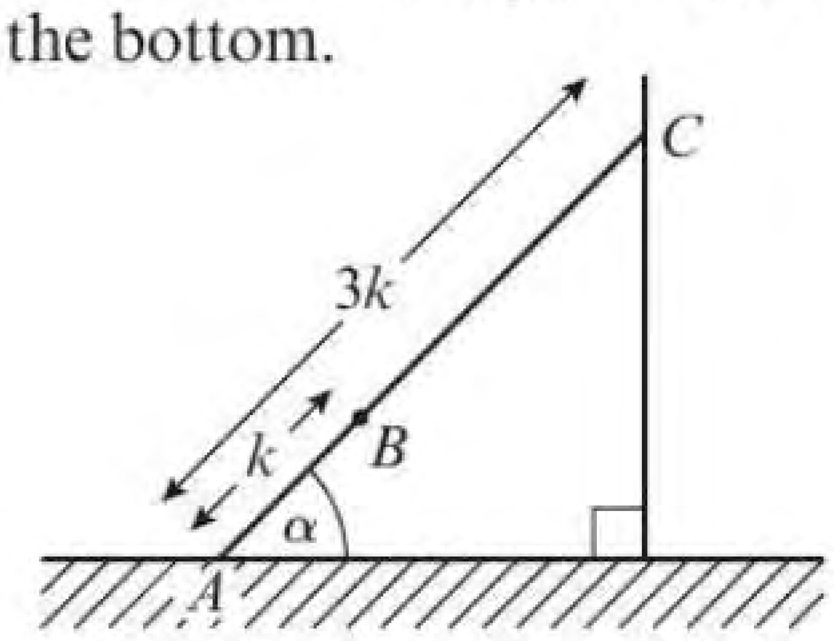
Then you need to resolve horizontally and vertically, and choose a point to take moments about. In this example, you need to find m, so you can't take moments about the top of the ladder. Take moments about the bottom of the ladder, and remember that:

Moment = force × **perpendicular distance**

You will need to use problem-solving skills throughout your exam – **be prepared!**

Now try this

A uniform ladder AC of mass m and length $3k$ rests on rough horizontal ground, leaning against a smooth horizontal wall. The angle between the ladder and the ground is α, where $\tan\alpha = \frac{3}{4}$. A builder of mass $5m$ stands on the ladder at a point B, a distance k from the bottom.



(a) Given that the builder and the ladder are in equilibrium, find the least possible value of the coefficient of friction between the ladder and the ground. **(8 marks)**

(b) State how you have used the fact that the ladder is uniform in your calculations. **(1 mark)**

When the value of the coefficient of friction is a minimum, the ladder will be in limiting equilibrium.