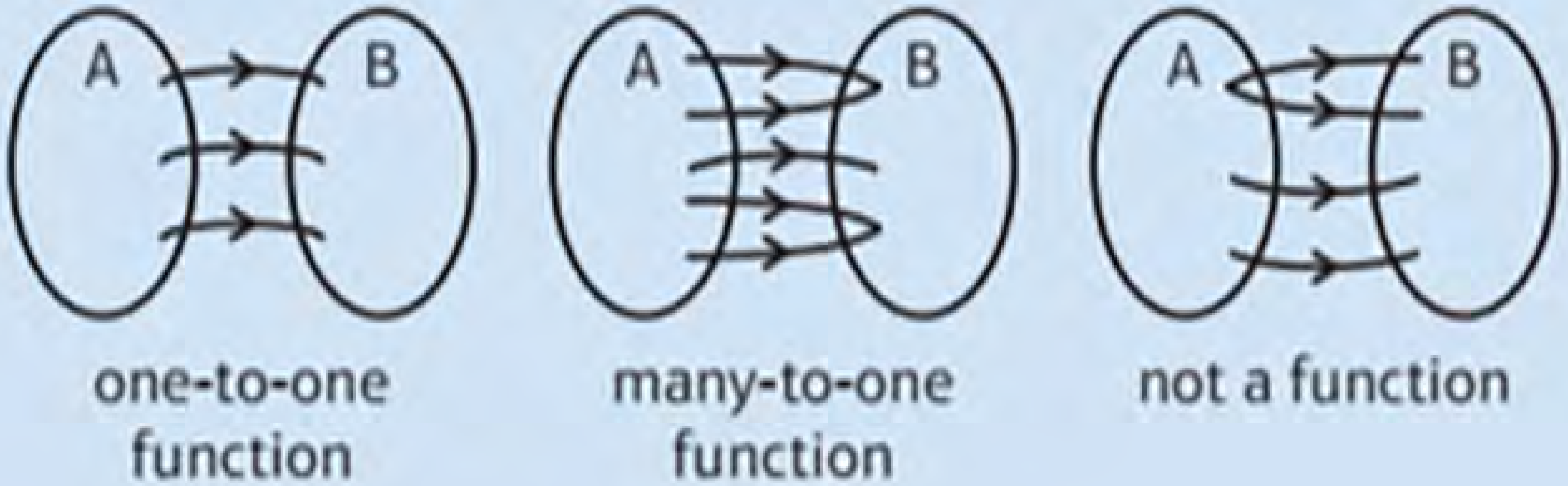
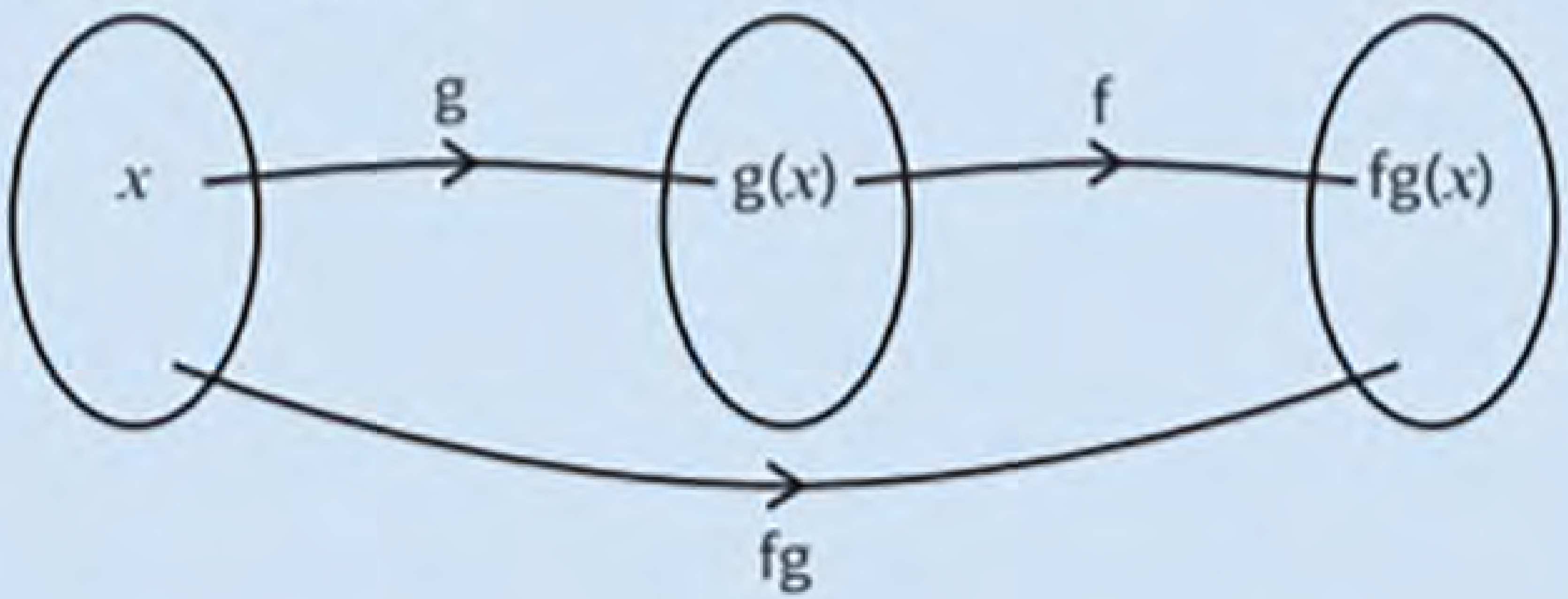
Summary of key points

3 A mapping is a **function** if every input has a distinct output. Functions can either be **one-to-one** or **many-to-one**.



4 $fg(x)$ means apply g first, then apply f.

$fg(x) = f(g(x))$



5 Functions $f(x)$ and $f^{-1}(x)$ are inverses of each other. $ff^{-1}(x) = x$ and $f^{-1}f(x) = x$.

6 The graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of each another in the line $y = x$.

7 The domain of $f(x)$ is the range of $f^{-1}(x)$.

8 The range of $f(x)$ is the domain of $f^{-1}(x)$.

Domain and range

A function maps numbers in its **domain** onto numbers in its **range**. Here is an example:

$$f(x) = \sqrt{x-2}, \quad x \geqslant 2$$

f is the **name** of the function. You can use any letter, but f and g are the most common.

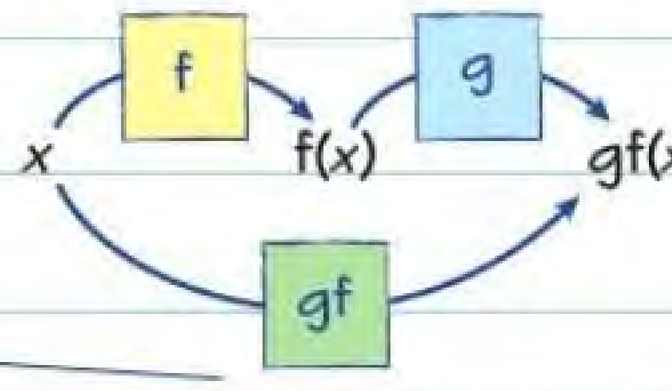
x is the **input**. You say 'f of x'. You can also write $f: x \rightarrow \sqrt{x-2}$ and say 'f maps x onto $\sqrt{x-2}$'.

This tells you what the function does to x.
$f(18) = \sqrt{18-2}$
$= \sqrt{16} = 4$

This is the **domain** of the function. The function is only defined for these **input** values. The **range** of this function is $f(x) \geqslant 0$. This tells you all the possible **output** values for the function.

Composite functions

If you apply two functions one after the other, you can write a **single function** which has the same effect as the two combined functions. This is called a **composite function**.



The function gf has the same effect as applying function f **then** applying function g.

The **order** is important. The function being applied **first** goes **closest** to the x.

Worked example

The functions f and g are defined by

$f: x \mapsto 1 + 4x^3, \quad x \in \mathbb{R}$

$g: x \mapsto 2 - \frac{1}{x}, \quad x \in \mathbb{R}, \quad 0 < x \leqslant 4$

(a) Show that the composite function gf is

$gf: x \mapsto \frac{1+8x^3}{1+4x^3}$ **(4 marks)**

$$gf(x) = 2 - \frac{1}{1+4x^3}$$
$$= \frac{2(1+4x^3) - 1}{1+4x^3}$$
$$= \frac{1+8x^3}{1+4x^3}$$

(b) Solve $gf(x) = 0$ **(2 marks)**

$\frac{1+8x^3}{1+4x^3} = 0$ so $1 + 8x^3 = 0$
$8x^3 = -1$
$x^3 = -\frac{1}{8}$
$x = -\frac{1}{2}$

gf(x) means you do f first, then g. To find an expression for gf(x) you need to substitute the **whole expression** for f(x) into the expression for g(x):

$g: x \mapsto 2 - \frac{1}{x}$

Substitute the whole expression for f(x) here.

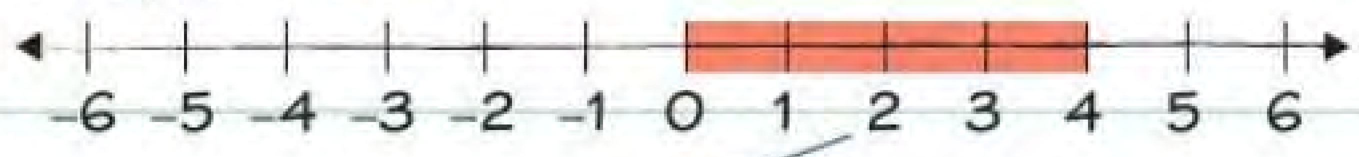
Real numbers

Most of the functions you encounter will be defined on some subset of the real numbers. Here are some domains of functions:

$x \in \mathbb{R}$

$x \in \mathbb{R}, x > 0$

$x \in \mathbb{R}, 0 < x \leqslant 4$



You can write this as (0, 4]. The square bracket shows that 4 is included.

Now try this

The functions f and g are defined by

$f: x \mapsto 3x + 2, \quad x \in \mathbb{R}$

$g: x \mapsto \frac{x}{x+2}, \quad x \in \mathbb{R}, \quad x \neq -2$

Don't worry about the domain. If you need to state a domain you will be told to do so in the question.

(a) Find $gf(-2)$. Answer **(2 marks)**

(b) Show that the composite function fg is

$fg: x \mapsto \frac{5x+4}{x+2}$ Answer **(4 marks)**

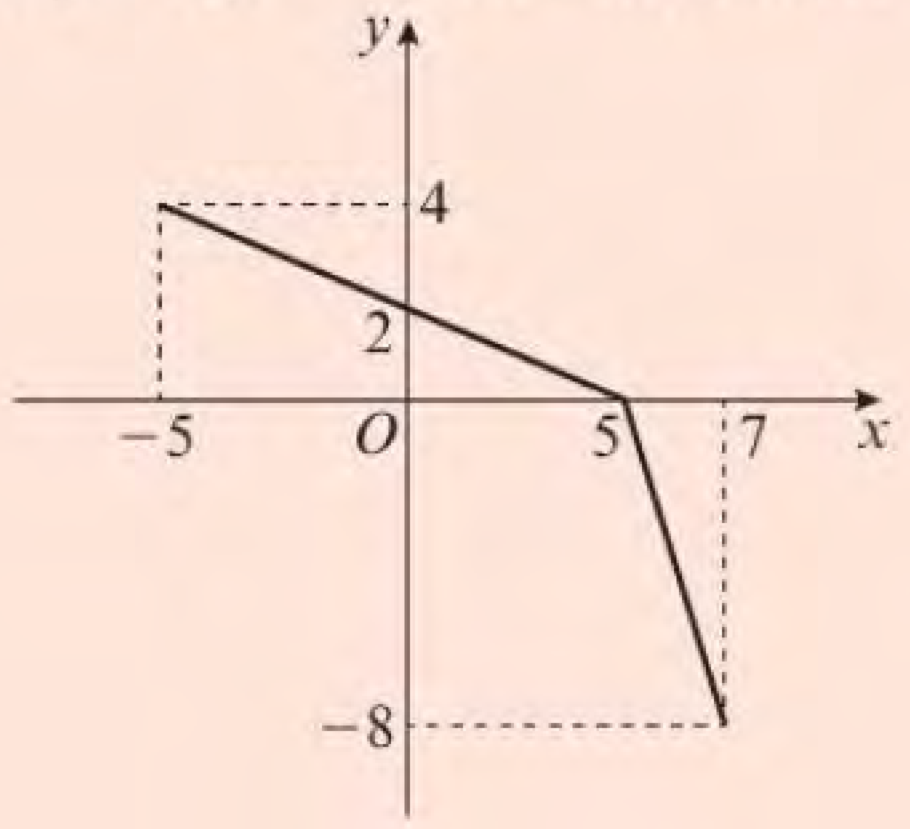
(c) Solve the equation $f(x) = [f(x)]^2$ **(4 marks)**
Answer

Graphs and range

You can use graphs of the form $y = f(x)$ or $y = g(x)$ to represent functions. The y-coordinates tell you the output values of the function (the **range**).

Worked example

The function f has domain $-5 \leqslant x \leqslant 7$. A sketch of the graph of $y = f(x)$ is shown below.



The range is the possible output values of the function. This is all the y-coordinates on the graph of the function:

Use the same type of inequality as the domain.

$-8 \leqslant f(x) \leqslant 4$

Use f(x) or y as the variable. Do not use x when writing the range.

(a) Write down the range of f. **(1 mark)**

$-8 \leqslant f(x) \leqslant 4$

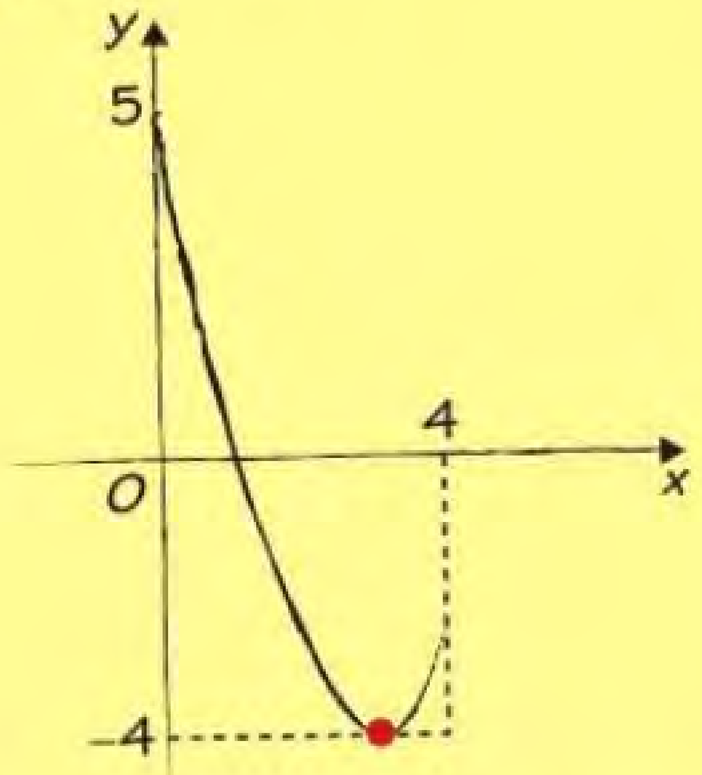
(b) Find ff(5). **(2 marks)**

$ff(5) = f(0) = 2$

ff(5) means f[f(5)]
When $x = 5$, $y = 0$, so $f(5) = 0$
When $x = 0$, $y = 2$, so $f(0) = 2$

Work out the value of the function near the edges of the domain. If you have a **graphing calculator** you could plot the graph to get an idea of its shape.

This is a quadratic function. You can find the minimum value by completing the square. Look at the edges of the domain to find the maximum.



Worked example

Find the range of the functions defined by:

(a) $f(x) = \frac{x+1}{x-3}$, $x > 5$ **(2 marks)**

When $x = 5$, $\frac{x+1}{x-3} = \frac{6}{2} = 3$

As $x \to \infty$, $\frac{x+1}{x-3} \to 1$ but is always > 1.

Range is $1 < f(x) < 3$

(b) $g(x) = x^2 - 6x + 5$, $x \in \mathbb{R}$, $0 \leqslant x \leqslant 4$ **(2 marks)**

$g(x) = (x - 3)^2 - 4$

So $g(x)$ has a minimum at $(3, -4)$.

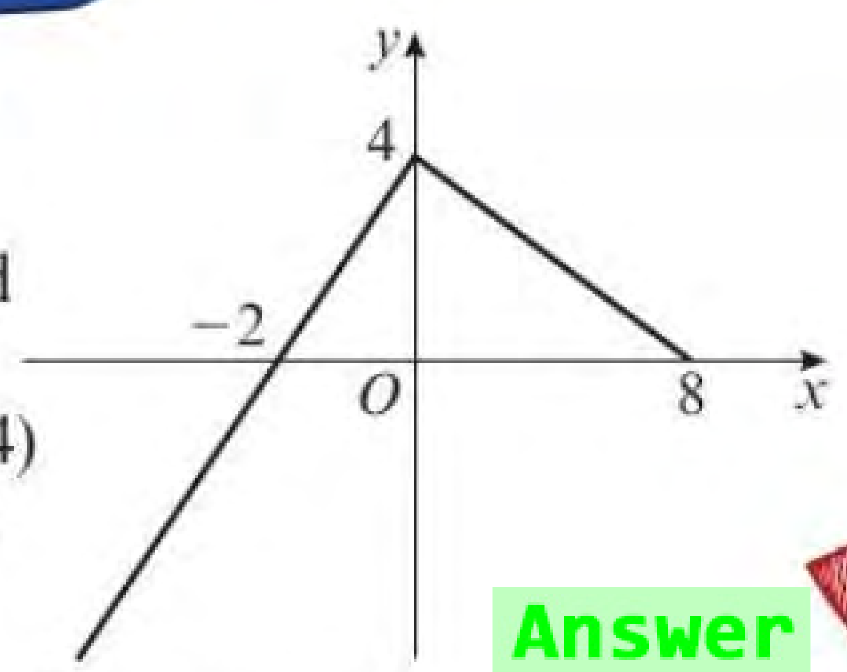
Maximum value of $g(x)$ occurs at $x = 0$.

$g(0) = 0^2 - 6 \times 0 + 5 = 5$

Range is $-4 \leqslant g(x) \leqslant 5$

Now try this

1 The function f has domain $-5 \leqslant x \leqslant 8$ and is linear from $(-5, -6)$ to $(0, 4)$ and from $(0, 4)$ to $(8, 0)$.



(a) Write down the range of f. **(1 mark)**

(b) Find ff(−1). **(2 marks)**

2 The function g is defined by

$$g : x \mapsto \frac{3x-7}{x^2-5x+6} - \frac{2}{x-3}, \quad x \in \mathbb{R}, \quad x > 3$$

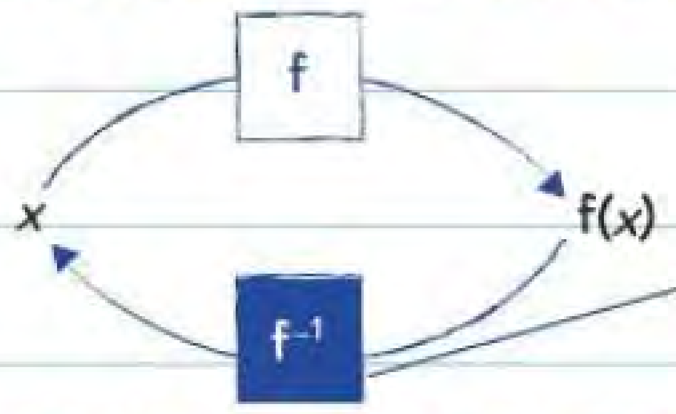
(a) Show that $g(x) = \frac{1}{x-2}$, $x > 3$ **(3 marks)**

(b) Find the range of g. **(2 marks)**

You need to use the fact that the two sections are linear to work out the values of the function between the given points.

Inverse functions

For a function f, the **inverse** of f is the function that **undoes** f. You write the inverse as f^{-1}. If you apply f then f^{-1}, you will end up back where you started.



If you apply f, then f^{-1} you have applied the **composite function** $f^{-1}f$. The output of $f^{-1}f$ is the **same** as the input. You can write: $f^{-1}f(x) = ff^{-1}(x) = x$

Finding the inverse

To find the inverse of a function given in the form $f(x) = \ldots$ you need to:

1. Write the function in the form $y = \ldots$
2. Rearrange to make x the subject
3. Swap any y's for x's and rewrite as $f^{-1}(x) = \ldots$

You aren't asked to state the domain here, so you don't need to include it in your answer.

Worked example

The function f is defined by:

$$f : x \mapsto \frac{2x+1}{4-x}, \quad x \in \mathbb{R}, \quad x \neq 4$$

Find $f^{-1}(x)$. **(3 marks)**

$$y = \frac{2x+1}{4-x}$$
$$y(4 - x) = 2x + 1$$
$$4y - xy = 2x + 1$$
$$4y - 1 = 2x + xy$$
$$4y - 1 = x(2 + y)$$
$$x = \frac{4y-1}{2+y}$$
$$f^{-1}(x) = \frac{4x-1}{2+x}$$

Worked example

The function g is defined by:

$$g : x \mapsto \frac{4}{x+1}, \quad x \in \mathbb{R}, \quad x > 3$$

(a) Find $g^{-1}(x)$. **(3 marks)**

$$y = \frac{4}{x+1}$$
$$y(x + 1) = 4$$
$$xy + y = 4$$
$$xy = 4 - y$$
$$x = \frac{4-y}{y} \quad \text{so} \quad g^{-1}(x) = \frac{4-x}{x}$$

(b) Find the domain of g^{-1}. **(2 marks)**

When $x = 3$, $\frac{4}{x+1} = 1$

As $x \to \infty$, $\frac{4}{x+1} \to 0$ but is always > 0.

So range of g is $0 < g(x) < 1$

So domain of g^{-1} is $0 < x < 1$

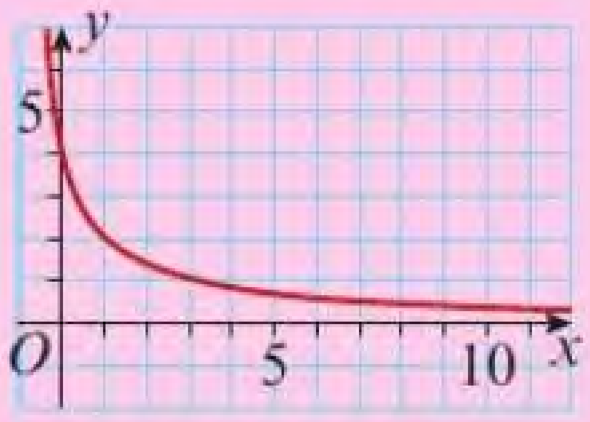
Golden rule

The **range** of a function is the **domain** of its **inverse**, and vice versa.

You can often use this rule to find the domain of an inverse function.

Problem solved!

Find the range of g to work out the domain of g^{-1}. If you have a graphing calculator you can sketch the graph of $y = \frac{4}{x+1}$ to see what it looks like.



You are interested in the y-values for $x > 3$.

You will need to use problem-solving skills throughout your exam – **be prepared!**

Now try this

The function h is defined by:

$$h : x \mapsto \frac{x+5}{x}, \quad x \in \mathbb{R}, \quad x > 1$$

Answer (a) Find $h^{-1}(x)$. **(3 marks)**

Answer (b) Write down the range of h^{-1}. **(1 mark)**

Answer (c) Find the domain of h^{-1}. **(2 marks)**

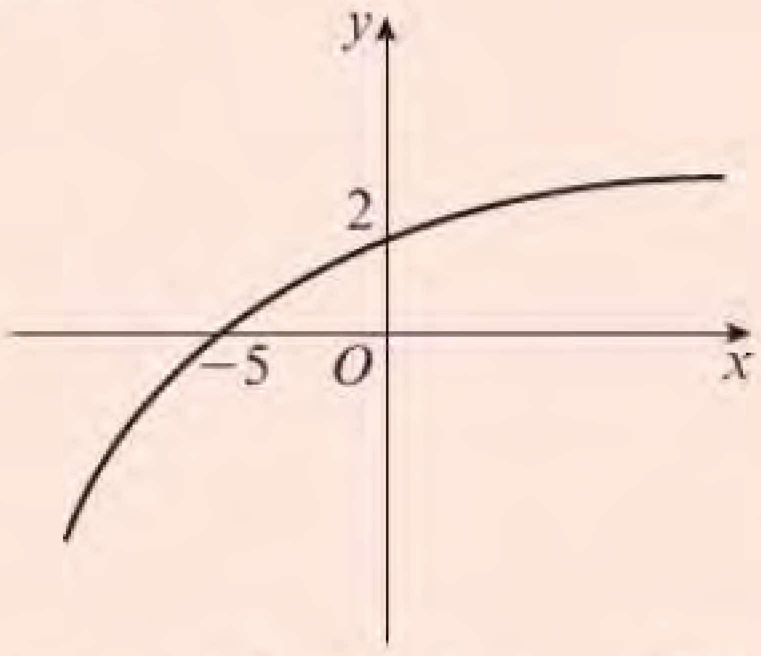
Write $y = \frac{x+5}{x}$ then rearrange to make x the subject. x appears twice on the right-hand side so you will need to factorise to get x on its own.

Inverse graphs

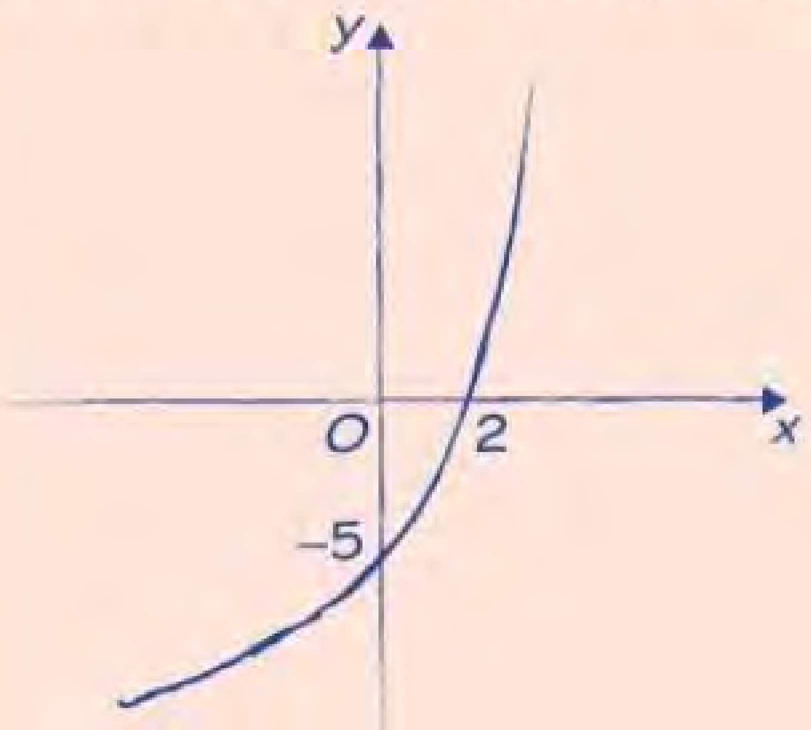
The graph of $y = f^{-1}(x)$ is the graph of $y = f(x)$ reflected in the line $y = x$.

Worked example

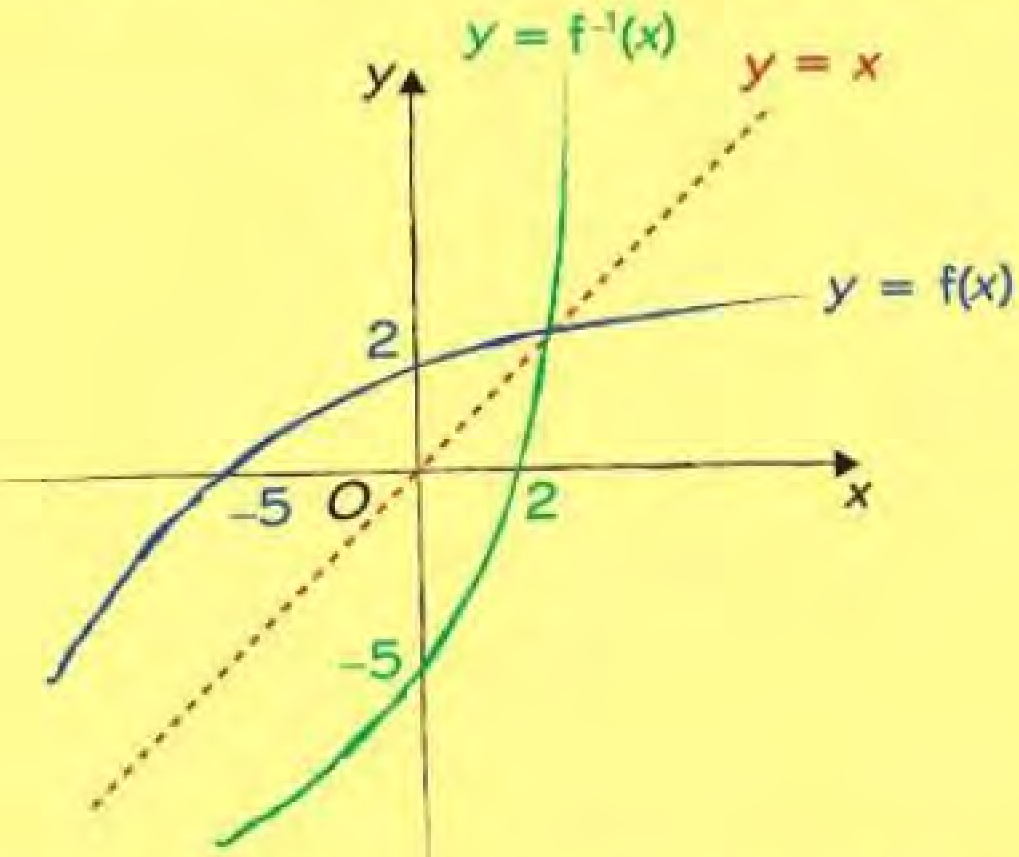
The diagram shows part of the curve with equation $y = f(x)$. The curve intersects the coordinate axes at $(-5, 0)$ and $(0, 2)$.



Sketch the curve with equation $y = f^{-1}(x)$. **(2 marks)**



You need to reflect the curve in the line $y = x$.



You can reflect a point in the line $y = x$ by swapping the x- and y-coordinates:
$(-5, 0) \rightarrow (0, -5)$ and $(0, 2) \rightarrow (2, 0)$

Worked example

The function h is defined by

$h : x \mapsto x^2 + 1, \quad x \in \mathbb{R}$

Explain why the function h does not have an inverse. **(1 mark)**

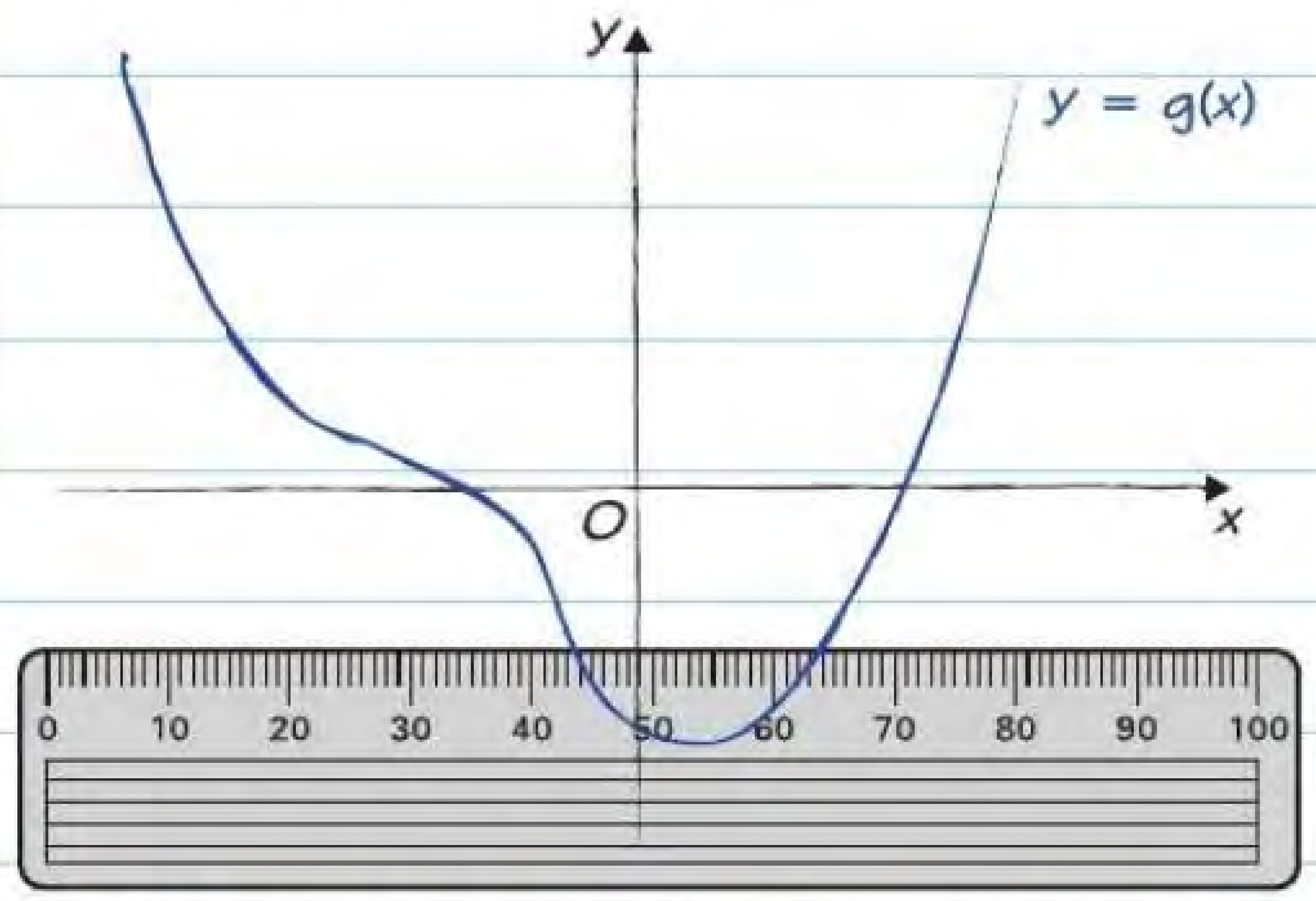
h is not a one-to-one function.

Both of these answers are correct:
- h is not a one-to-one function
- h is a many-to-one function.

You can also use the word 'mapping' instead of 'function' in your answer.

Existence of the inverse

The inverse of a function **only exists** if the function maps each point in its domain to a **unique** point in its range. This is sometimes called a **one-to-one** function. If f is a one-to-one function then a horizontal line drawn anywhere on the graph of $y = f(x)$ will never cross the graph more than **once**.



g is NOT a one-to-one function because a horizontal line crosses the graph of $y = g(x)$ more than once.

Now try this

The function f has domain $-8 \leqslant x \leqslant 4$ and is linear from $(-8, -5)$ to $(5, 0)$ and from $(5, 0)$ to $(6, 4)$. A sketch of the graph of $y = f(x)$ is shown.

Sketch the graph of $y = f^{-1}(x)$. Show the coordinates of the points corresponding to A and B. **(3 marks)**

Answer