Summary of key points

8 The Greek capital letter 'sigma' is used to signify a sum. You write it as $\sum$. You write limits on the top and bottom to show which terms you are summing.

9 A recurrence relation of the form $u_{n+1} = \text{f}(u_n)$ defines each term of a sequence as a function of the previous term.

10 A sequence is **increasing** if $u_{n+1} > u_n$ for all $n \in \mathbb{N}$.

A sequence is **decreasing** if $u_{n+1} < u_n$ for all $n \in \mathbb{N}$.

A sequence is **periodic** if the terms repeat in a cycle. For a periodic sequence there is an integer k such that $u_{n+k} = u_n$ for all $n \in \mathbb{N}$. The value k is called the **order** of the sequence.

Recurrence relations

A recurrence relation tells you how to find one term in a sequence from the **previous term**. You can use the recurrence relation and the **first term** to generate the sequence.

You can use any letter to represent a term in a sequence.

This number tells you the term number.

The **first term** is 5.

$$x_1 = 5$$

$$x_{n+1} = 2x_n + 3,\ n \geqslant 1$$

The **term-to-term** rule is 'multiply by 2 then add 3'.

The term numbers in a series are all positive integers.

So for this sequence:

$x_2 = 2x_1 + 3 = 2(5) + 3 = 13$

$x_3 = 2x_2 + 3 = 2(13) + 3 = 29$

... and so on.

Increasing, decreasing or periodic?

You need to be able to identify the following properties of some sequences:

- A sequence is **increasing** if $x_{n+1} > x_n$ for all $n \in \mathbb{N}$.
  2, 4, 6, 8... is an increasing sequence
- A sequence is **decreasing** if $x_{n+1} < x_n$ for all $n \in \mathbb{N}$.
  $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$... is a decreasing sequence
- A sequence is **periodic** if its terms repeat in a cycle. For a periodic sequence there exists an integer k such that $x_{n+k} > x_n$ for all $n \in \mathbb{N}$. The value of k is called the **period** or **order** of the sequence.
  −1, 0, 1, 0, −1, 0, 1, 0... is a periodic sequence with order 4

Worked example

A sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = k$, $a_{n+1} = 3a_n + 1, n \geqslant 1$, where k is a constant.

(a) Find the set of values of k for which the sequence is increasing. **(3 marks)**

$a_{n+1} > a_n$

$3a_n + 1 > a_n$

$a_n > -\frac{1}{2}$

So the sequence is increasing for all $k > -\frac{1}{2}$

If the sequence is increasing then $a_{n+1} > a_n$. Use the recurrence relation to rearrange this into the form $a_n > -\frac{1}{2}$. This must be true for all values of n, so it must be true for $a_1 = k$

(b) Given that $\sum_{r=1}^{3} a_r = 44$, show that $k = 3$ **(4 marks)**

$a_1 = k$ and $a_2 = 3k + 1$

$a_3 = 3a_2 + 1 = 3(3k + 1) + 1 = 9k + 4$

$\sum_{r=1}^{3} a_r = a_1 + a_2 + a_3$

$= k + (3k + 1) + (9k + 4) = 13k + 5$

$13k + 5 = 44$

$13k = 39$ so $k = 3$

Write an expression for $\sum_{r=1}^{3} a_r$ in terms of k.

Sigma notation

The Greek letter sigma ($\sum$) is used to show the **sum** of a set of terms in a sequence. The numbers above and below the $\sum$ tell you which terms to **add together**:

You are adding the first **four** terms of the sequence.

$$\sum_{r=1}^{4} u_r = u_1 + u_2 + u_3 + u_4$$

Substitute values of r from 1 up to 4, and add the terms.

Now try this

1 A sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = 5$, $a_{n+1} = 2a_n - 4, n \geqslant 1$

(a) Find the values of a_2 and a_3, and state whether the sequence is increasing, decreasing or neither. **(3 marks)** Answer

(b) Calculate the value of $\sum_{r=1}^{5} a_r$ **(3 marks)** Answer

2 A sequence $x_1, x_2, x_3, \ldots$ is defined by $x_1 = 2$, $x_{n+1} = px_n - 1, n \geqslant 1$ where p is a positive constant.

(a) Write down an expression for x_2 in terms of p. **(1 mark)** Answer

(b) Show that $x_3 = 2p^2 - p - 1$ **(2 marks)**

(c) Given that $x_3 = 9$, find the value of p. **(3 marks)** Answer

Answer

Write a quadratic equation and solve it by factorising. Remember that p is positive.

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