Summary of key points

1 In an **arithmetic sequence**, the difference between consecutive terms is constant.

2 The formula for the nth term of an arithmetic sequence is $\boldsymbol{u_n = a + (n - 1)d}$, where a is the first term and d is the common difference.

3 An arithmetic series is the sum of the terms of an arithmetic sequence.

The sum of the first n terms of an arithmetic series is given by $\boldsymbol{S_n = \frac{n}{2}(2a + (n - 1)d)}$, where where a is the first term and d is the common difference.

You can also write this formula as $\boldsymbol{S_n = \frac{n}{2}(a + l)}$, where l is the last term.

Arithmetic sequences

An arithmetic sequence is a sequence where the difference between consecutive terms is constant. You usually use a to represent the **first term** and d to represent the **common difference.** Here are two examples:

1 +3 +3 +3 +3

11, 14, 17, 20, 23 ...

$a = 11$, $d = 3$, nth term $= 11 + 3(n - 1)$

2 $-\frac{1}{2}$ $-\frac{1}{2}$ $-\frac{1}{2}$ $-\frac{1}{2}$

$1, \frac{1}{2}, 0, -\frac{1}{2}, -1 \ldots$

$a = 1$, $d = -\frac{1}{2}$, nth term $= 1 - \frac{1}{2}(n - 1)$

Finding the nth term

If an arithmetic sequence has first term a and common difference d, then the nth term, or **general term** is

$$a + (n - 1)d$$

Worked example

The first term of an arithmetic sequence is a and the common difference is d.

The 13th term of the sequence is 8 and the 16th term of the sequence is $12\frac{1}{2}$

(a) Write down two equations for a and d. **(2 marks)**

$a + (13 - 1)d = 8$ so $a + 12d = 8$ ①

$a + (16 - 1)d = 12\frac{1}{2}$ so $a + 15d = 12\frac{1}{2}$ ②

(b) Find the values of a and d. **(2 marks)**

② − ①: $3d = 4\frac{1}{2}$

$d = 1\frac{1}{2}$

Substitute $d = 1\frac{1}{2}$ into ①: $a + 12\left(1\frac{1}{2}\right) = 8$

$a + 18 = 8$

$a = -10$

You could also work out the common difference by writing down a few terms of the sequence:

$+1\frac{1}{2}$ $+1\frac{1}{2}$ $+1\frac{1}{2}$

…, 8, ?, ?, $12\frac{1}{2}$, …

There are three jumps between the 13th term and the 16th term. $12\frac{1}{2} - 8 = 4\frac{1}{2}$

So each jump is $4\frac{1}{2} \div 3 = 1\frac{1}{2}$

Solve the two equations simultaneously to find a and d. Number each equation to keep track of your working.

You can work in decimals or mixed numbers. You could give your answer as $a = -10$, $d = 1.5$

Now try this

1 The first term of an arithmetic sequence is a and the common difference is d.

The 9th term of the sequence is 3 and the 11th term of the sequence is −4

(a) Write down two equations for a and d. Answer **(2 marks)**

Answer (b) Find the values of a and d. **(2 marks)**

2 An arithmetic sequence has first term $p^2 + 1$ and common difference p, where $p > 0$

The 7th term of the sequence is 24

Work out the value of p. Give your answer in the form $a + b\sqrt{2}$, where a and b are integers. Answer **(5 marks)**

Use the information given to write a quadratic equation involving p. You need the answer in surd form so solve your equation by completing the square. There is more about this on page 4.

3 Beth is saving money for a deposit on a house. In the first month she saves £300. Each month she increases the amount she saves by £20

(a) Show that in the fifth month Beth saves £380 Answer **(1 mark)**

(b) Find an expression in terms of n for the amount Beth saves in the nth month. Answer **(2 marks)**

Arithmetic series

In a series, the terms are always **added together**. A series is arithmetic if its terms have a common difference, like 8 + 10 + 12 + 14 + …. Here is another example:

$a + (n - 1)d = 20$
$70 - 5(n - 1) = 20$
$n = 11$

$-5 \quad -5$

$70 + 65 + 60 + \ldots + 20$

You could use the formula on the right to work out the sum of this series.
Use $a = 70$, $d = -5$ and $n = 11$. The sum is:

$\frac{1}{2}(11)[2(70) + (11 - 1)(-5)] = \frac{1}{2}(11)(90) = 495$

Sum to *n* terms

If an **arithmetic** series has first term a and common difference d, then the **sum** of the first n terms is $\frac{1}{2}n[2a + (n - 1)d]$.

This formula is given in the booklet. But you also need to know its **proof** and this is **not** in the formulae booklet. Look at the first Worked example below to see the proof in action.

Worked example

An arithmetic series has first term a and common difference d. Prove that the sum of the first n terms of the series, S, is

$\frac{1}{2}n[2a + (n - 1)d]$ **(4 marks)**

$S = a + (a + d) + \ldots + [a + (n - 1)d]$ ①
$S = [a + (n - 1)d] + \ldots + (a + d) + a$ ②
① + ②:
$2S = [2a + (n - 1)d] + \ldots + [2a + (n - 1)d]$
$2S = n[2a + (n - 1)d]$
$S = \frac{1}{2}n[2a + (n - 1)d]$

To prove this you need to write the sum **forwards** and **backwards**.
You can then add pairs of terms to get an expression for $2S$:

$a + [a + (n - 1)d] = 2a + (n - 1)d$
$(a + d) + [a + (n - 2)d] = 2a + (n - 1)d$
$(a + 2d) + [a + (n - 3)d] = 2a + (n - 1)d$
… and so on.

Every pair of terms adds up to $2a + (n - 1)d$, and there are n pairs, so $2S = n[2a + (n - 1)d]$

Worked example

An arithmetic series has first term 13 and common difference 4
The sum of the first n terms of the sequence is 175

(a) Show that n satisfies $2n^2 + 11n - 175 = 0$ **(3 marks)**

$a = 13$, $d = 4$, $S = 175$, $S_n = \frac{1}{2}n[2a - (n - 1)d]$
$\frac{1}{2}n[26 + 4(n - 1)] = 175$
$13n + 2n^2 - 2n = 175$
$2n^2 + 11n - 175 = 0$

(b) Hence find the value of n. **(3 marks)**

$(2n + 25)(n - 7) = 0$
$n = -\frac{25}{2}$ or $n = 7$

Substitute the values of a and d into the formula to get an expression for the sum **in terms of** n. Set this expression equal to 175 and simplify to get a quadratic equation in n.

Solve your quadratic equation by factorising. n is the number of terms in the series, so it must be a **positive integer**. This means you can ignore the negative solution.

Now try this

1 An arithmetic series has first term −3 and common difference d.
The sum of the first 20 terms of the series is 320
Find the value of d. Answer **(4 marks)**

2 Prove that the sum of the first n natural numbers is $\frac{1}{2}n(n + 1)$
Answer **(3 marks)**

You can't use the formula because it says 'prove'. Try to apply the technique in the first Worked example.

Modelling with series

You can model a variety of real-life situations using **sequences** or **series**.

Worked example

Nisha is buying a car. She pays for it monthly over 12 months. The garage offers two repayment plans. Both plans form an arithmetic sequence.

Plan A: First month payment £X
Payments decrease by £$2Y$ each month

Plan B: First month payment £$(X - 1100)$
Payments decrease by £Y each month

(a) Show that the **total** amount paid under Plan A for the 12-month period is £$(12X - 132Y)$ **(2 marks)**

$a = X$, $d = -2Y$, $n = 12$

$$S_n = \frac{1}{2}n[2a + (n-1)d]$$
$$= \frac{1}{2}(12)[2X - 2Y(12-1)]$$
$$= 6(2X - 22Y)$$
$$= 12X - 132Y$$

(b) For the 12-month period, the **total** paid is the same for both plans. Find the value of Y. **(4 marks)**

For Plan B, $S_n = \frac{1}{2}n[2a + (n-1)d]$
$$= \frac{1}{2}(12)[2(X - 1100) - Y(12-1)]$$
$$= 12X - 66Y - 13200$$
$$12X - 66Y - 13200 = 12X - 132Y$$
$$66Y = 13200$$
$$Y = 200$$

(c) Under Plan A, Nisha's final payment would be £300. Work out the value of X. **(3 marks)**

$$X + (12-1)(-400) = 300$$
$$X - 4400 = 300$$
$$X = 4700$$

The payments **decrease**, so the values of d in your arithmetic sequences will be **negative**.
For Plan A: $a = X$, $d = -2Y$ and $n = 12$
For Plan B: $a = (X - 1100)$, $d = -Y$ and $n = 12$

Write down the formula for the sum of the first n terms of an arithmetic series. This formula is given in the formulae booklet. Then substitute for a, d and n and simplify your expression.

Write your expression for the total paid under Plan B, make it equal to $12X - 132Y$ and solve an equation to find Y.

Problem solved!

The monthly payments **decrease** by $2Y$ each month, so $d = -400$. Make sure you use the formula for the nth term, not for the sum. Write an equation and solve it to find the value of X.

You will need to use problem-solving skills throughout your exam – **be prepared!**

Now try this

Hanna is planning a training regime. On the first day she does 25 sit-ups. Each day she increases the number of sit-ups by 6.

(a) Find an expression for the number of sit-ups Hanna does on the nth day of her regime. Answer **(2 marks)**

(b) Show that the total number of sit-ups she had done after n days is $3n^2 + 22n$ Answer **(3 marks)**

(c) On the nth day of her regime, Hanna does 79 sit-ups. How many sit-ups has she done in total in the n days of her regime? **(4 marks)**
Answer