Summary of key points

3 For positive values of a and b,

- $a \sin x \pm b \cos x$ can be expressed in the form $R \sin (x \pm \alpha)$
- $a \cos x \pm b \sin x$ can be expressed in the form $R \cos (x \mp \alpha)$

with $R > 0$ and $0 < \alpha < 90°$ $\left(\text{or } \frac{\pi}{2}\right)$

$a\cos\theta \pm b\sin\theta$

You can use the addition formulae to write expressions of the form $a\cos\theta \pm b\sin\theta$ in the form $R\cos(\theta \mp \alpha)$ or $R\sin(\theta \pm \alpha)$. This can help you solve harder trig equations.

If you use these rules, be careful with the signs. You will usually be able to use **positive** values of a and b.

Golden rules

1. $a\cos\theta \pm b\sin\theta = R\cos(\theta \mp \alpha)$
2. $a\sin\theta \pm b\cos\theta = R\sin(\theta \pm \alpha)$

where $R = \sqrt{a^2 + b^2}$ and $\alpha = \arctan\left(\frac{b}{a}\right)$

Worked example

(a) Express $5\cos x - 3\sin x$ in the form $R\cos(x + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$ **(4 marks)**

$R = \sqrt{5^2 + 3^2} = \sqrt{34}$

$\alpha = \arctan\left(\frac{3}{5}\right) = 0.5404...$

$5\cos x - 3\sin x = \sqrt{34}\cos(x + 0.5404...)$

(b) Hence, or otherwise, solve the equation

$5\cos x - 3\sin x = 4$

for $0 \leqslant x \leqslant 2\pi$, giving your answers to 2 decimal places. **(5 marks)**

$\sqrt{34}\cos(x + 0.5404...) = 4$

$\cos(x + 0.5404...) = \frac{4}{\sqrt{34}}$

$x + 0.5404... = 0.8148...$

$x = 0.8148... - 0.5404...$

$= 0.2744...$

or $x + 0.5404... = 2\pi - 0.8148...$

$= 5.4683...$

$x = 5.4683... - 0.5404...$

$= 4.9279...$

$x = 0.27, 4.93$ (2 d.p.)

If you don't want to learn the rules above, you can use the addition formulae from page 82:

$5\cos x - 3\sin x = R\cos x\cos\alpha - R\sin x\sin\alpha$

Equate coefficients of $\cos x$: $5 = R\cos\alpha$ ①

Equate coefficients of $\sin x$: $3 = R\sin\alpha$ ②

①^2 + ②^2: $R^2(\cos^2\alpha + \sin^2\alpha) = 5^2 + 3^2$

$R = \sqrt{5^2 + 3^2}$

② ÷ ①: $\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{3}{5}$

$\alpha = \arctan\left(\frac{3}{5}\right)$

Be careful. In part (b) of this question, the principal value from your calculator will not give you an answer in the required range.

Problem solved!

Pay attention to these three Rs when solving trig equations in your exam:

- **Rounding** – Don't round any values until the end of your calculation. Learn how to use the 'STORE' or 'MEMORY' functions on your calculator so you can use unrounded values, or write down values to at least 4 decimal places.
- **Radians** – Look at the range to decide whether you should be working in degrees or radians. Make sure your calculator is in the correct mode.
- **Range** – Check that all your solutions are within the specified range, and check that you have found every possible solution within that range.

You will need to use problem-solving skills throughout your exam – **be prepared!**

Now try this

1 (a) Express $3\sin 2\theta + 2\cos 2\theta$ in the form $R\sin(2\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$ Answer **(4 marks)**

(b) Hence, or otherwise, solve the equation

$3\sin 2\theta + 2\cos 2\theta + 1 = 0$

for $0 \leqslant \theta \leqslant \pi$, giving your answers to 2 decimal places. **(5 marks)** Answer

2 The function f is defined by

$y = \text{f} : x \mapsto \sqrt{3}\cos x + \sin x$

(a) Given that $\text{f}(x) = R\cos(x - \alpha)$, where $R > 0$ and $0 < \alpha < 90°$, find the value of R and the value of α. Answer **(4 marks)**

(b) Hence sketch the graph of $y = \text{f}(x)$ for $0 \leqslant x \leqslant 360°$, showing clearly the coordinates of any maxima or minima, and any points where the graph meets the coordinate axes. Answer **(5 marks)**

Trig modelling

Tides, circular motion, pendulums and springs are just a few examples of real-life situations that can be modelled using trigonometric functions.

Worked example

(a) Express $2\sin\theta - 1.5\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the value of α to 4 decimal places. **(3 marks)**

$R = \sqrt{2^2 + 1.5^2} = 2.5$

$\alpha = \arctan\left(\frac{1.5}{2}\right) = 0.64350...$

$2\sin\theta - 1.5\cos\theta = 2.5\sin(\theta - 0.6435...)$

(b) (i) Find the maximum value of $2\sin\theta - 1.5\cos\theta$

(ii) Find the value of θ, for $0 \leqslant \theta < \pi$, at which this maximum occurs. **(3 marks)**

(i) 2.5

(ii) $\sin(\theta - 0.6435...) = 1$

$\theta - 0.6435... = \frac{\pi}{2}$

$\theta = 2.2143$ (4 d.p.)

Tom models the height of sea water, H metres, on a particular day by the equation $H = 6 + 2\sin\left(\frac{4\pi t}{25}\right) - 1.5\cos\left(\frac{4\pi t}{25}\right)$, $0 \leqslant t < 12$, where t is the number of hours after midday.

(c) Calculate the maximum value of H predicted by this model and the value of t, to 2 decimal places, when this maximum occurs. **(3 marks)**

$H = 6 + 2.5\sin\left(\frac{4\pi t}{25} - 0.6435...\right)$

$H_{max} = 8.5$

Occurs when $\frac{4\pi t}{25} = 2.2143...$

so $t = 4.41$ (2 d.p.)

(d) Calculate, to the nearest minute, the times when the height of sea water is predicted, by this model, to be 7 metres. **(6 marks)**

$6 + 2.5\sin\left(\frac{4\pi t}{25} - 0.6435...\right) = 7$

$\sin\left(\frac{4\pi t}{25} - 0.6435...\right) = 0.4$

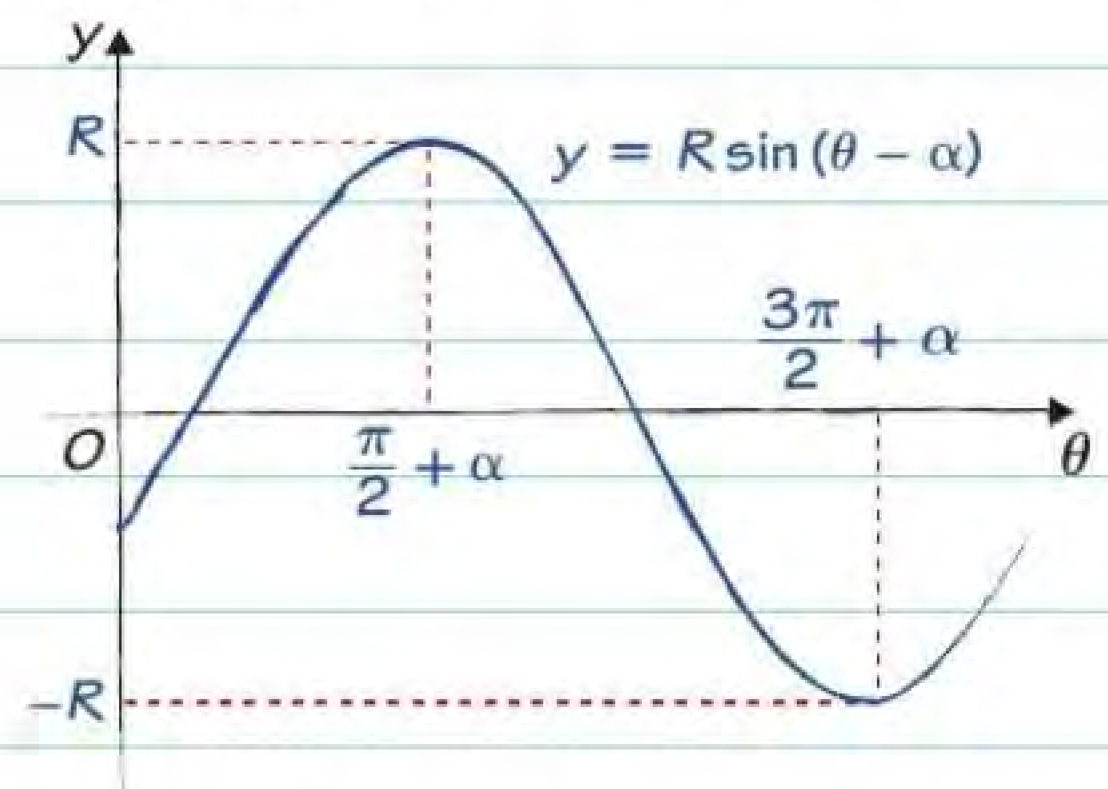
$\frac{4\pi t}{25} - 0.6435... = 0.4115...$ or $\pi - 0.4115...$

$t = 2.0988...$ = 2.06 pm

$t = 6.7115...$ = 6.43 pm

Maximum and minimum

A function in the form $R\cos(\theta \pm \alpha)$ or $R\sin(\theta \pm \alpha)$ has a maximum value of R and a minimum value of $-R$.



The maximum value of $R\sin(\theta - \alpha)$ occurs when $\sin(\theta - \alpha) = 1$.

This occurs when $\theta - \alpha = \frac{\pi}{2}, \frac{5\pi}{2}, ...$

There will often be a connection between different parts of the same exam question. Look for similarities between the complicated equation in part (c) and the expression in part (a).

Part (d) says 'times' so make sure you find all possible solutions for the range of the model. Remember to convert to minutes. Multiply the decimal part by 60: $0.0988... \times 60 = 5.9...$

Now try this

The displacement, d cm, of a pendulum at time t seconds is modelled by the equation

$$d = 4\cos 1.2t + 2\sin 1.2t, \quad t > 0$$

(a) Given that $d = R\cos(1.2t - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$, find the value of R and the value of α. Answer **(3 marks)**

(b) Write down the maximum displacement of the pendulum. Answer **(3 marks)**

(c) Find the times in the interval $0 < t < 5$ when the displacement of the pendulum is 0. Answer **(2 marks)**