Summary of key points

1 For small angles, measured in radians:

- $\sin x \approx x$
- $\cos x \approx 1 - \frac{1}{2}x^2$

Differentiating sin x and cos x

You need to be able to differentiate sin and cos **from first principles**. For a reminder about differentiation from first principles, have a look at page 35.

Worked example

Prove, from first principles, that the derivative of $\cos x$ is $-\sin x$. You may assume that as $h \to 0$, $\frac{\sin h}{h} \to 1$ and $\frac{\cos h - 1}{h} \to 0$. **(5 marks)**

Let $f(x) = \cos x$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0}\frac{\cos(x+h) - \cos(x)}{h}$$
$$= \lim_{h\to 0}\frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h}$$
$$= \lim_{h\to 0}\left(\left(\frac{\cos h - 1}{h}\right)\cos x - \left(\frac{\sin h}{h}\right)\sin x\right)$$

Since $\frac{\cos h - 1}{h} \to 0$ and $\frac{\sin h}{h} \to 1$ the limit tends to $0 \times \cos x - 1 \times \sin x = -\sin x$

So $f'(x) = -\sin x$ as required.

Problem solved!

1. Write out the rule for differentiation from first principles – this is given in the formulae booklet.
2. Substitute $f(x) = \cos x$.
3. Use the addition formula on $\cos(x + h)$.
4. Rearrange the expression in the limit to get the expressions $\frac{\cos h - 1}{h}$ and $\frac{\sin h}{h}$ on their own – this will allow you to use the conditions given in the question.
5. State how you are using the conditions given in the question.
6. Write down what you have proved.

You will need to use problem-solving skills throughout your exam – **be prepared!**

Small angle approximations

The proof shown above makes use of small angle approximations for sin and cos. When x is small and measured in radians:

$\sin x \approx x$ $\cos x \approx 1 - \frac{1}{2}x^2$ $\tan x \approx x$

You can use these approximations to deduce the limits given in the proof above. When h is small:

$\frac{\sin h}{h} \approx \frac{h}{h} = 1$ and

$\frac{\cos h - 1}{h} \approx \frac{1 - \frac{1}{2}h^2 - 1}{h} = -\frac{1}{2}h$

This tends to 0 as $h \to 0$

Worked example

Show that, for small values of θ measured in radians, $\frac{1 - 4\cos\theta + \sin\theta}{3 + \tan 2\theta} \approx -1$ **(4 marks)**

$$\frac{1 - 4\cos\theta + \sin\theta}{3 + \tan 2\theta} \approx \frac{1 - 4(1 - \frac{1}{2}\theta^2) + \theta}{3 + 2\theta}$$
$$= \frac{2\theta^2 + \theta - 3}{3 + 2\theta}$$
$$= \frac{(2\theta + 3)(\theta - 1)}{2\theta + 3}$$
$$= \theta - 1$$

$\theta - 1 \to -1$ as $\theta \to 0$

If θ is small then you can assume that small multiples of θ such as 2θ and 3θ are also small.

Now try this

1 (a) Given that $f(x) = \sin x$, show that

$$f'(x) = \lim_{h\to 0}\left(\left(\frac{\cos h - 1}{h}\right)\sin x + \left(\frac{\sin h}{h}\right)\cos x\right)$$
Answer **(3 marks)**

(b) Hence prove that $f'(x) = \cos x$ **(2 marks)** Answer

2 Given that θ is small and measured in radians, show that $\frac{1 - \cos 3\theta}{\tan\theta \sin\theta} \approx \frac{9}{2}$ **(4 marks)**
Answer

3 $f(x) = \cos x$

Prove, from first principles, that $f'\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$ **(5 marks)**
Answer

In question 3, follow the same steps that are shown in the worked example above, but replace x with $\frac{\pi}{3}$.