Summary of key points

1. $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$ $\quad \int e^x \, dx = e^x + c$ $\quad \int \frac{1}{x} \, dx = \ln|x| + c$

   $\int \cos x \, dx = \sin x + c$ $\quad \int \sin x \, dx = -\cos x + c$ $\quad \int \sec^2 x = \tan x + c$

   $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$ $\quad \int \operatorname{cosec}^2 x \, dx = -\cot x + c$ $\quad \int \sec x \tan x \, dx = \sec x + c$

2. $\int f'(ax + b) \, dx = \frac{1}{a} f(ax + b) + c$

3. Trigonometric identities can be used to integrate expressions. This allows an expression that cannot be integrated to be replaced by an identical expression that can be integrated.

4. To integrate expressions of the form $\int k \frac{f'(x)}{f(x)} \, dx$, try $\ln|f(x)|$ and differentiate to check, and then adjust any constant.

5. To integrate an expression of the form $\int k f'(x)(f(x))^n \, dx$, try $(f(x))^{n+1}$ and differentiate to check, and then adjust any constant.

Integrating standard functions

Make sure you are really familiar with these four integrals. They're not given in the formulae booklet and you will use them a lot in your exam.

Trigonometric functions

1. $\int \cos x \, dx = \sin x + c$
2. $\int \sin x \, dx = -\cos x + c$

Remember that indefinite integrations always include a **constant of integration**, c.

Exponential functions

3. $\int e^x dx = e^x + c$
4. $\int \left(\frac{1}{x}\right) dx = \ln|x| + c$

You use a modulus sign when you integrate $\frac{1}{x}$, because $\ln x$ only exists for positive values of x.

Integrating f′(ax + b)

This is a useful integration rule to learn:

$$\int f'(ax+b)\,dx = \frac{1}{a}f(ax+b) + c$$

This is an application of the chain rule 'in reverse'. There are more examples of this on the next page.

Worked example

(a) Find $\int \cos 2x \, dx$ **(2 marks)**

$\int \cos 2x \, dx = \frac{1}{2}\sin 2x + c$

(b) Find $\int \frac{6}{3x+1} dx$ **(2 marks)**

$\int \frac{6}{3x+1} dx = 2\ln|3x+1| + c$

$\int \frac{6}{3x+1} dx = 6\int \frac{1}{3x+1} dx$

This is a **definite integration**. You don't need to use a constant of integration.

Worked example

Show that $\int_0^1 \sqrt{2-x}\, dx = \frac{4\sqrt{2}-2}{3}$ **(3 marks)**

$$\int_0^1 \sqrt{2-x}\,dx = \int_0^1 (2-x)^{\frac{1}{2}} dx$$
$$= \left[-\frac{2}{3}(2-x)^{\frac{3}{2}}\right]_0^1$$
$$= \left(-\frac{2}{3}(1)^{\frac{3}{2}}\right) - \left(-\frac{2}{3}(2)^{\frac{3}{2}}\right)$$
$$= -\frac{2}{3} + \frac{2}{3}(2\sqrt{2}) = \frac{4\sqrt{2}-2}{3}$$

You can use the rule for integrating f(ax + b) to integrate any of these functions:

$f(x) = \frac{1}{ax+b}$ $\quad f(x) = (ax+b)^n$

$f(x) = \sin(ax+b)$ $\quad f(x) = \cos(ax+b)$

Remember to multiply by $\frac{1}{a}$ and leave $ax + b$ inside the function. You can always check your answers by differentiating.

Now try this

1 (a) Find $\int 4\sin 2x \, dx$ Answer **(2 marks)**

(b) Find $\int e^{\frac{x}{6}} dx$ Answer **(2 marks)**

2 Find the exact value of $\int_0^{\frac{\pi}{2}} 2\sin\frac{1}{2}\theta \, d\theta$ Answer **(3 marks)**

3 Find $\int \frac{2}{1-4x} dx$ Answer **(2 marks)**

$\int \frac{2}{1-4x} dx = 2\int \frac{1}{1-4x} dx$

Use the rule for integrating functions of the form $f'(ax+b)$, and be careful with the signs.

Had a look ☐ Nearly there ☐ Nailed it! ☐

Reverse chain rule

You can use the chain rule **in reverse** to integrate some expressions. Here are two useful examples. If you can spot these integrations you can save time in your exam.

1 $\int f'(x)[f(x)^n]\,dx = \frac{1}{n+1}[f(x)]^{n+1} + c$

For example: ($n = 10$)

$\int 2x(x^2 + 1)^{10}\,dx = \frac{1}{11}(x^2+1)^{11} + c$

$f'(x) = 2x$, $f(x) = x^2 + 1$

2 $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$

For example:

$\int \frac{6x^2}{2x^3 - 1}\,dx = \ln|2x^3 - 1| + c$

$f'(x) = 6x^2$, $f(x) = 2x^3 - 1$

Worked example

(a) Find $\int \frac{x}{5x^2+1}\,dx$ **(2 marks)**

$\int \frac{x}{5x^2+1}\,dx = \frac{1}{10}\int \frac{10x}{5x^2+1}\,dx$

$= \frac{1}{10}\ln|5x^2+1| + c$

(b) Find $\int x^2\sqrt{1-4x^3}\,dx$ **(3 marks)**

$\int x^2\sqrt{1-4x^3}\,dx = \int x^2(1-4x^3)^{\frac{1}{2}}\,dx$

Try $y = (1-4x^3)^{\frac{3}{2}}$

$\frac{dy}{dx} = -12x^2 \times \frac{3}{2}(1-4x^3)^{\frac{1}{2}} = -18x^2(1-4x^3)^{\frac{1}{2}}$

So $\int x^2\sqrt{1-4x^3}\,dx = -\frac{1}{18}(1-4x^3)^{\frac{3}{2}} + c$

Adjusting constants

If you can guess the **form** of the integral, but can't work out what constant to multiply it by, then you can:

1. Differentiate your guess for the integral.
2. Compare it with the original expression.
3. Adjust the constant if necessary.

For part (b) it's hard to see what constant to take outside the integral. Instead, write a guess for the integral as '$y = \ldots$'

Now find $\frac{dy}{dx}$ and compare it with $x^2(1-4x^3)^{\frac{1}{2}}$.

It is -18 times the original integrand, so divide y by -18.

Problem solved!

Watch out for common standard derivatives, especially when trigonometric functions are involved. You might need to use results from the formulae booklet to spot the form of the integral.

f(x)	f′(x)
$\sec kx$	$k\sec kx\tan kx$

You will need to use problem-solving skills throughout your exam – **be prepared!**

Worked example

Find $\int 2\tan x\sec^5 x\,dx$ **(3 marks)**

$\int 2\tan x\sec^5 x\,dx = \int 2\tan x\sec x(\sec x)^4\,dx$

Try $y = \sec^5 x$

$\frac{dy}{dx} = \sec x\tan x \times 5\sec^4 x$

$= 5\tan x\sec^5 x$

So $\int 2\tan x\sec^5 x\,dx = \frac{2}{5}\sec^5 x + c$

Now try this

1 Find $\int \frac{x+1}{x^2+2x-3}\,dx$ Answer **(3 marks)**

2 Find $\int \frac{2\cos 3x}{\sin 3x}\,dx$ Answer **(3 marks)**

3 The function f is defined by

$f : x \mapsto \frac{e^x}{2e^x - 1}, \quad x \geqslant 0$

Find the area enclosed by the curve with equation $y = f(x)$, the line $x = 1$ and the coordinate axes. Give your answer correct to 3 decimal places. Answer **(4 marks)**

Try $y = \ln(2e^x - 1)$

Identities in integration

You can use trigonometric identities to simplify integrations. If you're not sure which identity to use, have a look at the standard integrals in the formulae booklet. See if you can write the integral in terms of one of these functions.

You can write the integral in terms of $\sec^2$ using the identity

$\sec^2\theta \equiv 1 + \tan^2\theta$

Have a look at page 80 for identities involving $\sec^2$ and cosec^2.

Worked example

Find $\int \tan^2\left(\frac{x}{2}\right) dx$ **(2 marks)**

$$\int \tan^2\left(\frac{x}{2}\right) dx = \int\left(\sec^2\left(\frac{x}{2}\right) - 1\right) dx$$

$$= 2\tan\left(\frac{x}{2}\right) - x + c$$

Use this result from the formulae booklet with $k = \frac{1}{2}$:

$f(x)$	$\int f(x)\,dx$
$\sec^2 kx$	$\frac{1}{k}\tan kx$

$\sin^2 x$ and $\cos^2 x$

You can integrate $\sin^2 x$ and $\cos^2 x$ using the double angle formulae for cos:

1. $\cos 2A \equiv 2\cos^2 A - 1$
2. $\cos 2A \equiv 1 - 2\sin^2 A$

Have a look at page 83 for a reminder about these identities.

Worked example

Show that $\int_{\frac{\pi}{4}}^{\pi} \sin^2 x\,dx = \frac{2 + 3\pi}{8}$ **(5 marks)**

$$\int_{\frac{\pi}{4}}^{\pi} \sin^2 x\,dx = \int_{\frac{\pi}{4}}^{\pi}\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$$

$$= \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_{\frac{\pi}{4}}^{\pi}$$

$$= \left(\frac{\pi}{2} - \frac{1}{4}\sin 2\pi\right) - \left(\frac{\pi}{8} - \frac{1}{4}\sin\frac{\pi}{2}\right)$$

$$= \frac{3\pi}{8} + \frac{1}{4}$$

$$= \frac{2 + 3\pi}{8}$$

Use identity 2 from the box above to write $\sin^2 x$ in terms of $\cos 2x$. The question says 'show that', so make sure you clearly show the integrated function before substituting your limits and evaluating the integral.

Now try this

1 Find the exact value of $\int_0^{\frac{\pi}{12}} \sin 3x \cos 3x\,dx$

Answer **(5 marks)**

2 (a) By writing $\sin 7x$ as $\sin(4x + 3x)$ and by writing $\sin x$ as $\sin(4x - 3x)$, show that $\sin 7x + \sin x \equiv 2\sin 4x \cos 3x$

Answer **(4 marks)**

(b) Hence, or otherwise, find

Answer $\int \sin 4x \cos 3x\,dx$ **(2 marks)**

Use the identity

$\sin 2A \equiv 2\sin A\cos A$

with $A = 3x$

Use the addition formulae on page 82 and add together the two expressions. Part (b) says 'Hence' so you can save a lot of time by using your answer to part (a) to simplify the integral.