Summary of key points

9 The area bounded by two curves can be found using integration:

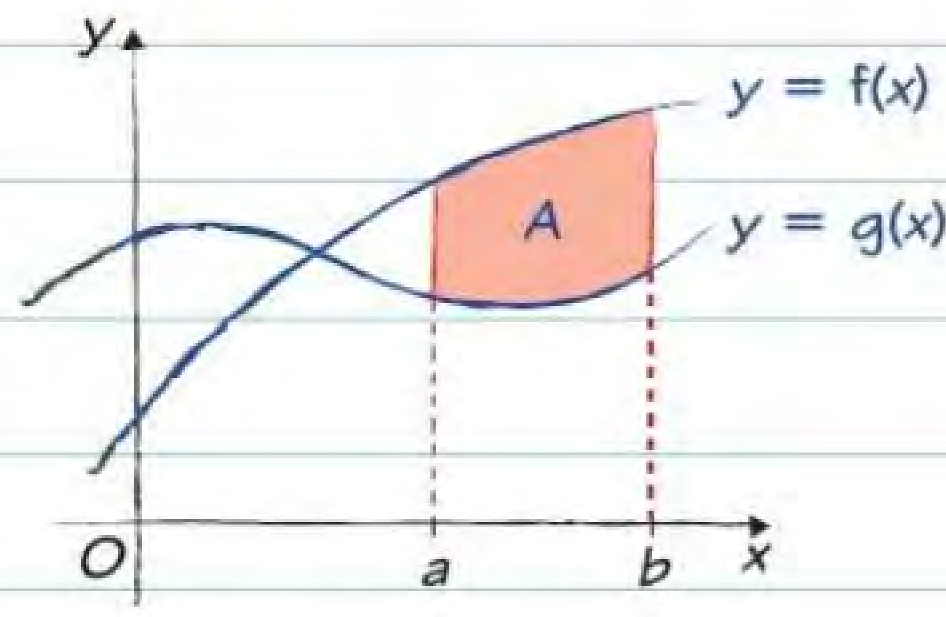
Area of $R = \int_a^b (\mathrm{f}(x) - \mathrm{g}(x))\,\mathrm{d}x = \int_a^b \mathrm{f}(x)\,\mathrm{d}x - \int_a^b \mathrm{g}(x)\,\mathrm{d}x$

Area between two curves

You can find the area between two curves using integration. In some cases you can use a formula to find the area between two curves. If $f(x) \geqslant g(x)$ for all values of x in the range $a \leqslant x \leqslant b$, then

$$A = \int_a^b (f(x) - g(x))\,dx$$

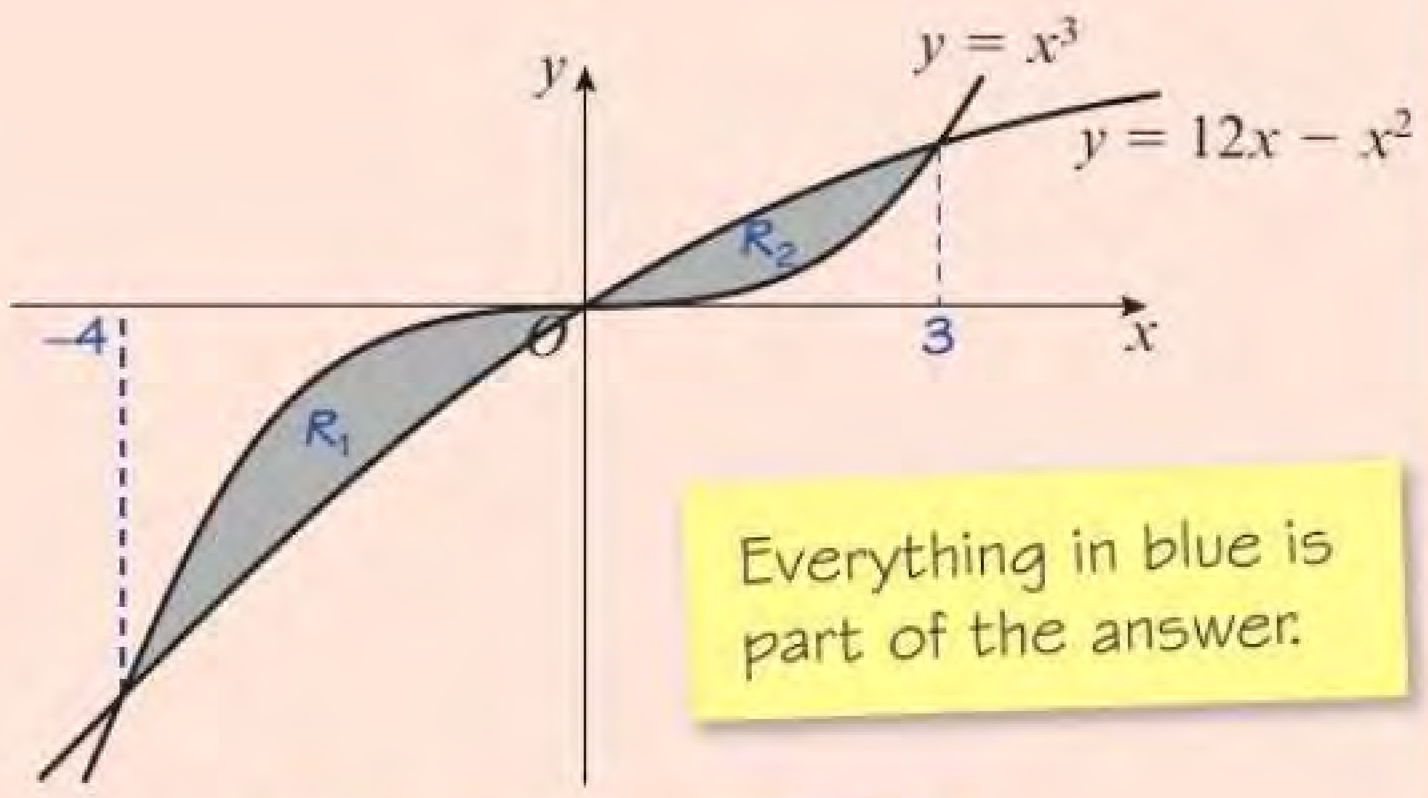
But watch out. If the curves intersect then you need to find the point of intersection and do two separate calculations.



Worked example

The diagram shows the curves with equations $y = x^3$ and $y = 12x - x^2$.

Find the total area of the shaded region bounded by the two curves. **(6 marks)**



$x^3 = 12x - x^2$

$x^3 + x^2 - 12x = 0$

$x(x + 4)(x - 3) = 0$

Curves intersect when $x = -4$, $x = 0$, $x = 3$

$$\text{Area } R_1 = \int_{-4}^{0} (x^3 - (12x - x^2))\,dx$$
$$= \left[\tfrac{1}{4}x^4 - 6x^2 + \tfrac{1}{3}x^3\right]_{-4}^{0}$$
$$= 0 - \left(\tfrac{1}{4}(-4)^4 - 6(-4)^2 + \tfrac{1}{3}(-4)^3\right)$$
$$= \frac{160}{3}$$

$$\text{Area } R_2 = \int_0^3 (12x - x^2 - x^3)\,dx$$
$$= \left[6x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right]_0^3$$
$$= 6(3)^2 - \tfrac{1}{3}(3)^3 - \tfrac{1}{4}(3)^4 - 0 = \frac{99}{4}$$

$$\text{Total area} = \frac{160}{3} + \frac{99}{4} = \frac{937}{12}$$

Everything in blue is part of the answer.

Problem solved!

The curves **intersect** at the origin. To use the formula given above, you need to consider the two regions separately. Label the regions R_1 and R_2, and work out the points of intersection, then use $A = \int_a^b (f(x) - g(x))\,dx$ twice.

You will need to use problem-solving skills throughout your exam – **be prepared!**

Now try this

The diagram shows the curves with equations $y = \sin x$ and $y = \sqrt{3}\cos x$ in the range $0 \leqslant x \leqslant \frac{3\pi}{2}$. The curves intersect at the points P and Q.

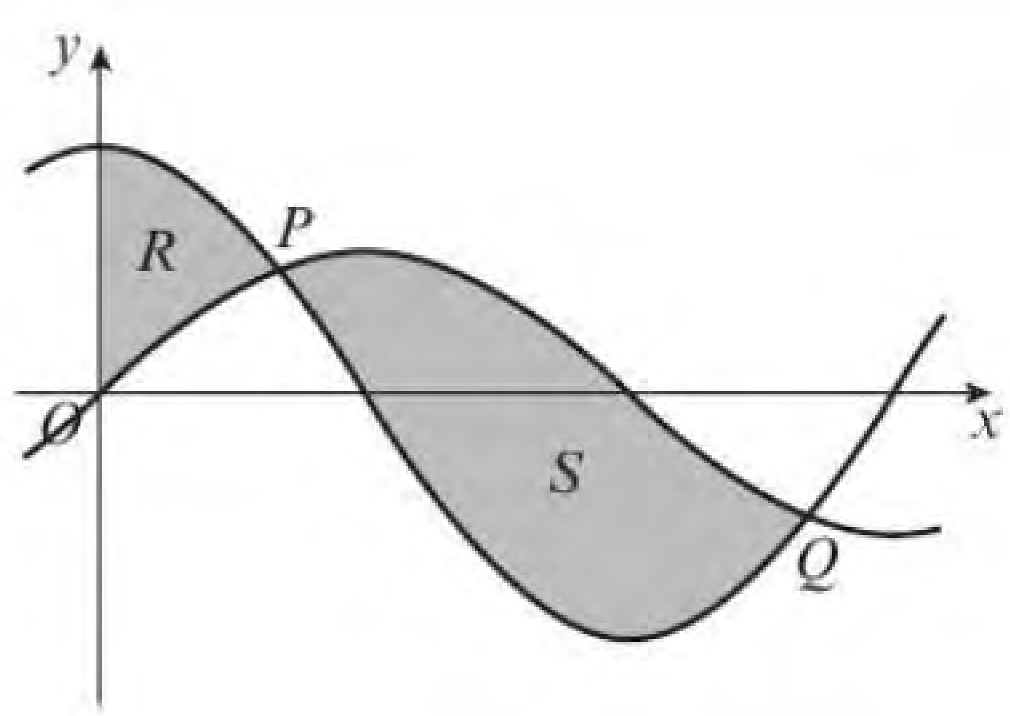
Answer

(a) Find the coordinates of P and Q. **(3 marks)**

(b) Find the area of the finite region R bounded by the curves and the y-axis. Answer **(5 marks)**

(c) Find the area of the finite region S bounded by the curves between P and Q. **(7 marks)**

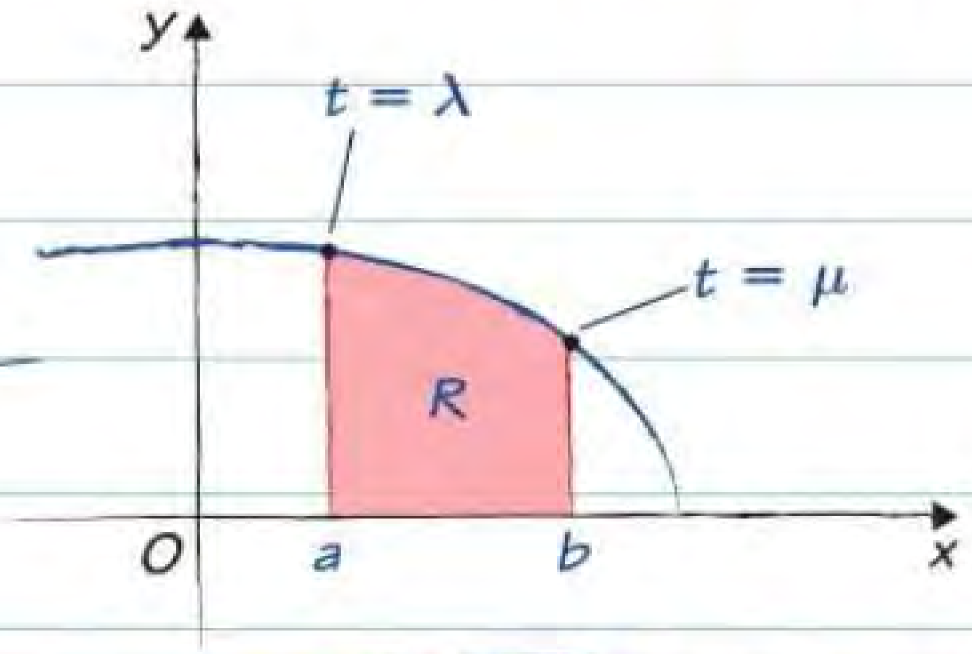
Answer



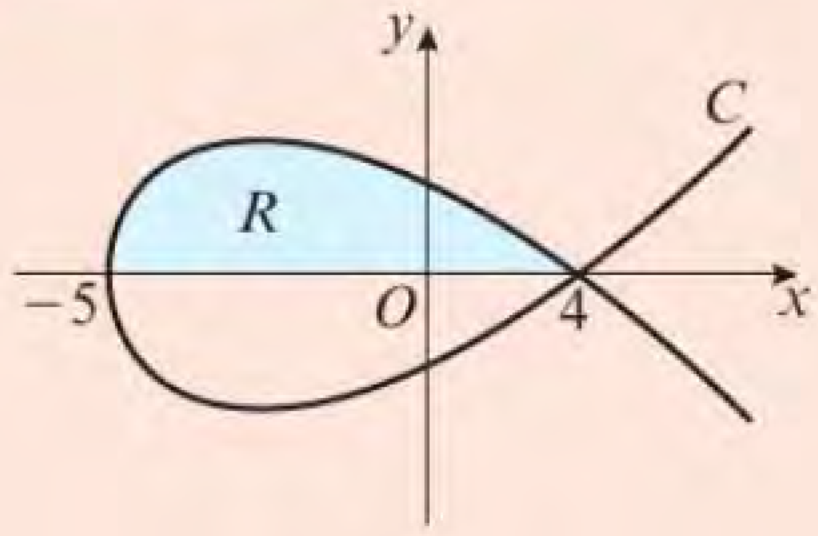
Areas and parametric curves

You can use integration to find the area under a **parametric** curve. It's almost always easier to integrate **with respect to the parameter**, t. For a reminder about parametric equations have a look at page 86.

$$R = \int_{x=a}^{x=b} y\,dx = \int_{t=\lambda}^{t=\mu} y\frac{dx}{dt}\,dt$$



Worked example



The curve C with parametric equations

$$x = 3t^2 - 5 \qquad y = t(3 - t^2)$$

cuts the x-axis at $(-5, 0)$ and $(4, 0)$.

The region R is enclosed by the curve and the x-axis. Use integration to find the area of R in the form $k\sqrt{3}$ where k is a constant. **(5 marks)**

When $x = -5$, $t = 0$

When $x = 4$, $t = \sqrt{3}$

$\frac{dx}{dt} = 6t$ so $dx = 6t\,dt$

$$R = \int_{x=-5}^{x=4} y\,dx = \int_{t=0}^{t=\sqrt{3}} t(3 - t^2)(6t)\,dt$$

$$= \int_{t=0}^{t=\sqrt{3}} (18t^2 - 6t^4)\,dt$$

$$= \left[6t^3 - \frac{6}{5}t^5\right]_0^{\sqrt{3}}$$

$$= 6(\sqrt{3})^3 - \frac{6}{5}(\sqrt{3})^5$$

$$= \frac{36}{5}\sqrt{3}$$

Problem solved!

When you evaluate a definite integral with respect to the parameter t, you must **transform the limits** into values of t. The first steps of your working should be writing down the values of t at the limits, and finding $\frac{dx}{dt}$ so you can write down the relationship between dx and dt.

You will need to use problem-solving skills throughout your exam – **be prepared!**

When you've been practising lots of complicated integrations, don't get caught out by an easy one. The expression in t is just a polynomial, so multiply out and integrate.

You can write your limits as '$x = \ldots$' or '$t = \ldots$' so you know which is which. Make sure your limits **match** your operator (dx or dt) before evaluating the integral.

For part (b) you will need to use this result from the formulae booklet:

$f(x)$	$\int f(x)\,dx$
$\sec kx$	$\frac{1}{k}\ln\lvert\sec kx + \tan kx\rvert$

Now try this

The diagram shows the curve with parametric equations

$$x = \tan t \qquad y = 2\cos t - 1 \qquad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

(a) Find the coordinates of the points A and B where the curve cuts the coordinate axes. **(3 marks)**

Answer

The region R is bounded by the curve, and the x- and y-axes.

(b) Use integration to find the area of R. Give your answer correct to 3 decimal places. **(4 marks)**

Answer