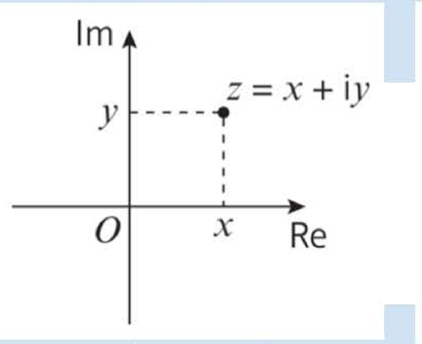
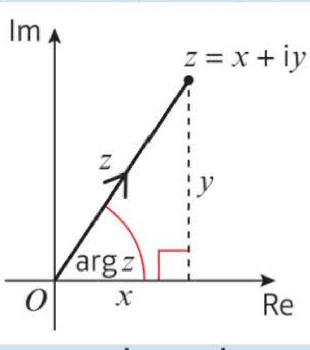
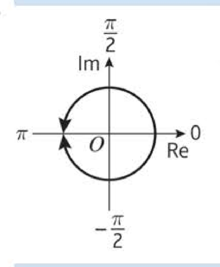
Summary of key points

1 You can represent complex numbers on an **Argand diagram**. The x-axis on an Argand diagram is called the **real axis** and the y-axis is called the **imaginary axis**. The complex number $z = x + iy$ is represented on the diagram by the point $P(x, y)$, where x and y are Cartesian coordinates.







The argument θ of any complex number is usually given in the range $-\pi < \theta \leqslant \pi$. This is sometimes referred to as the **principal argument**.

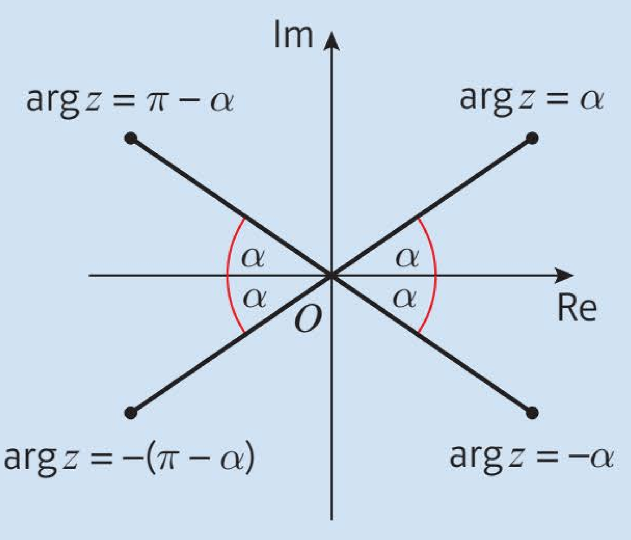
2 The complex number $z = x + iy$ can be represented as the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ on an Argand diagram.

3 The **modulus** of a complex number, $|z|$, is the distance from the origin to that number on an Argand diagram. For a complex number $z = x + iy$, the modulus is given by $|z| = \sqrt{x^2 + y^2}$.

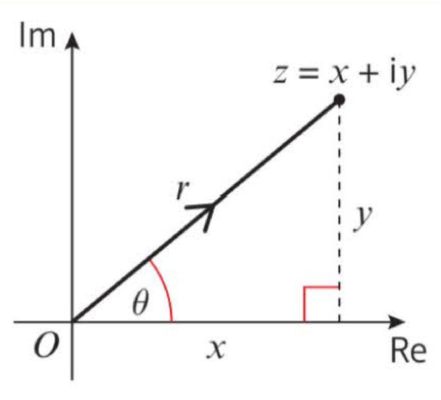
4 The **argument** of a complex number, arg z, is the angle between the positive real axis and the line joining that number to the origin on an Argand diagram. For a complex number $z = x + iy$, the argument, θ, satisfies $\tan\theta = \frac{y}{x}$

5 Let α be the positive acute angle made with the real axis by the line joining the origin and z.

- If z lies in the first quadrant then $\arg z = \alpha$.
- If z lies in the second quadrant then $\arg z = \pi - \alpha$.
- If z lies in the third quadrant then $\arg z = -(\pi - \alpha)$.
- If z lies in the fourth quadrant then $\arg z = -\alpha$.



6 For a complex number z with $|z| = r$ and $\arg z = \theta$, the modulus–argument form of z is $z = r(\cos\theta + i\sin\theta)$



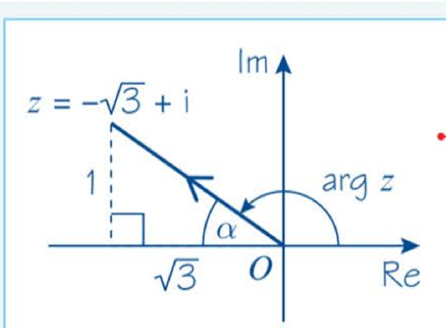
From the right-angled triangle, $x = r\cos\theta$ and $y = r\sin\theta$.

$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$

This formula works for a complex number in any quadrant of the Argand diagram. The argument, θ, is usually given in the range $-\pi < \theta \leqslant \pi$, although the formula works for any value of θ measured anticlockwise from the positive real axis.

Example 6

Express $z = -\sqrt{3} + i$ in the form $r(\cos\theta + i\sin\theta)$, where $-\pi < \theta \leqslant \pi$.



Sketch the Argand diagram, showing the position of the number.

Here z is in the second quadrant, so the required argument is $\pi - \alpha$.

$r = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$

$\theta = \arg z = \pi - \arctan\left(\frac{1}{\sqrt{3}}\right) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

Find r and θ.

Therefore, $z = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

Apply $z = r(\cos\theta + i\sin\theta)$.

7 For any two complex numbers z_1 and z_2,

- $|z_1 z_2| = |z_1||z_2|$
- $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

To prove these results, consider z_1 and z_2 in modulus–argument form:

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Multiplying these numbers together, you get

$$\begin{aligned} z_1 z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2) \\ &= r_1 r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) \\ &= r_1 r_2((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)) \\ &= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \end{aligned}$$

Links The last step of this working makes use of the trigonometric addition formulae:
$\sin(A \pm B) \equiv \sin A\cos B \pm \cos A\sin B$
$\cos(A \pm B) \equiv \cos A\cos B \mp \sin A\sin B$
← Pure Year 2, Section 7.1

Dividing z_1 by z_2 you get

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \\ &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)} \\ &= \frac{r_1(\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - i^2\sin\theta_1\sin\theta_2)}{r_2(\cos\theta_2\cos\theta_2 - i\cos\theta_2\sin\theta_2 + i\sin\theta_2\cos\theta_2 - i^2\sin\theta_2\sin\theta_2)} \\ &= \frac{r_1((\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2))}{r_2(\cos^2\theta_2 + \sin^2\theta_2)} \\ &= \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)) \end{aligned}$$

Links The last step of this working makes use of the trigonometric addition formulae together with the identity $\sin^2\theta + \cos^2\theta \equiv 1$
← Pure Year 1, Section 10.3

Express $z_1 z_2$ in the form $x + iy$.

Rewrite z_2 in the form $z_2 = r(\cos\theta + i\sin\theta)$:

Watch out z_2 is not initially given in modulus–argument form.

$\cos\left(-\frac{2\pi}{5}\right) = \cos\frac{2\pi}{5}$ and $\sin\left(-\frac{2\pi}{5}\right) = -\sin\frac{2\pi}{5}$

Use $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$

$z_2 = 3\left(\cos\left(-\frac{2\pi}{5}\right) + i\sin\left(-\frac{2\pi}{5}\right)\right)$

z_2 is now in modulus–argument form.

$z_1 z_2 = 2\left(\cos\frac{\pi}{15} + i\sin\frac{\pi}{15}\right) \times 3\left(\cos\left(-\frac{2\pi}{5}\right) + i\sin\left(-\frac{2\pi}{5}\right)\right)$

$= 2 \times 3\left(\cos\left(\frac{\pi}{15} - \frac{2\pi}{5}\right) + i\sin\left(\frac{\pi}{15} - \frac{2\pi}{5}\right)\right)$

Apply the result $z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$

$= 6\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$

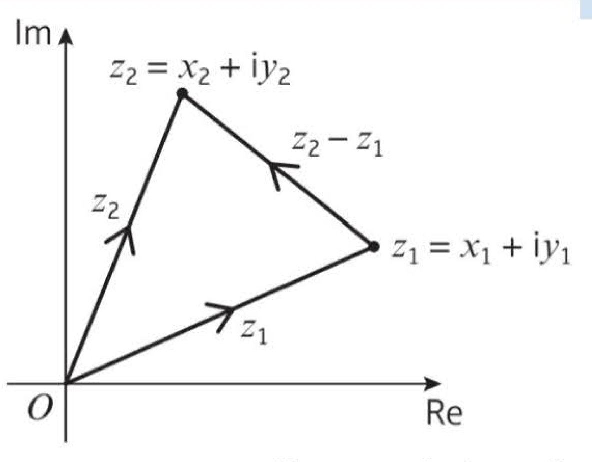
$= 6\left(\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right)$

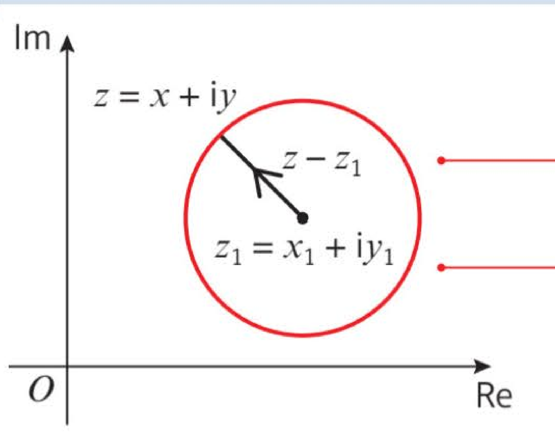
Apply $\cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$ and $\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$= 3 - 3\sqrt{3}i$

8 For two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, $|z_2 - z_1|$ represents the distance between the points z_1 and z_2 on an Argand diagram.

9 Given $z_1 = x_1 + iy_1$, the locus of points z on an Argand diagram such that $|z - z_1| = r$, or $|z - (x_1 + iy_1)| = r$, is a circle with centre (x_1, y_1) and radius r.





Locus of points.

Every point z, on the circumference of the circle, is a distance of r from the centre of the circle.

You can derive a Cartesian form of the equation of a circle from this form by squaring both sides:

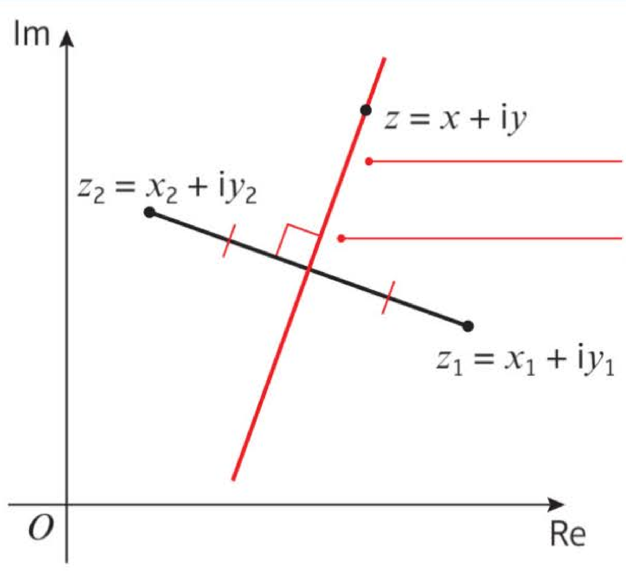
$|z - z_1| = r$

$|(x - x_1) + i(y - y_1)| = r$

$(x - x_1)^2 + (y - y_1)^2 = r^2$

Since $|p + qi| = \sqrt{p^2 + q^2}$

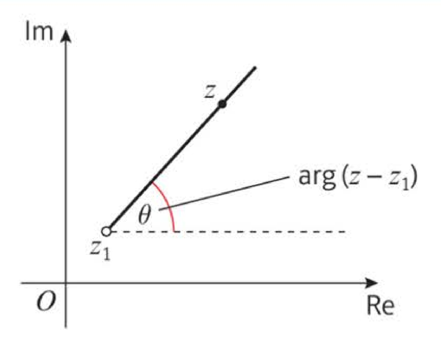
10 Given $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the locus of points z on an Argand diagram such that $|z - z_1| = |z - z_2|$ is the perpendicular bisector of the line segment joining z_1 and z_2.



Locus of points.

Every point z on the line is an equal distance from points z_1 and z_2.

11 Given $z_1 = x_1 + iy_1$, the locus of points z on an Argand diagram such that $\arg(z - z_1) = \theta$ is a half-line from, but not including, the fixed point z_1 making an angle θ with a line from the fixed point z_1 parallel to the real axis.



Notation A **half-line** is a straight line extending from a point infinitely in one direction only. You use an open circle to show that the point z, is not included in the locus.

You can find the Cartesian equation of the half-line corresponding to $\arg(z - z_1) = \theta$ by considering how the argument is calculated:

$\arg(z - z_1) = \theta$

$\arg((x - x_1) + i(y - y_1)) = \theta$

$\frac{y - y_1}{x - x_1} = \tan\theta$

$y - y_1 = \tan\theta\,(x - x_1)$

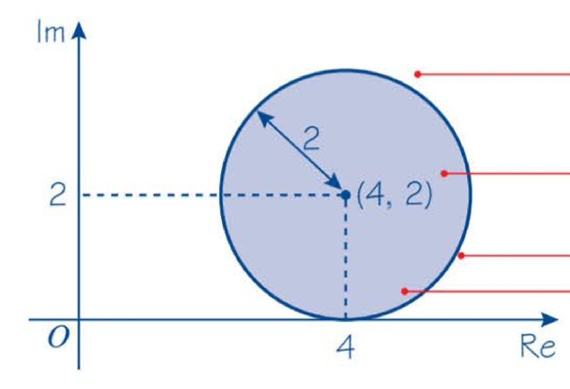
θ is a fixed angle so $\tan\theta$ is a constant.

This is the equation of a straight line with gradient $\tan\theta$ passing through the point (x_1, y_1).

Watch out The locus is the half-line so you need to give a suitable range of values for x.

You can use complex numbers to represent regions on an Argand diagram.

a i $|z - 4 - 2i| \leqslant 2$

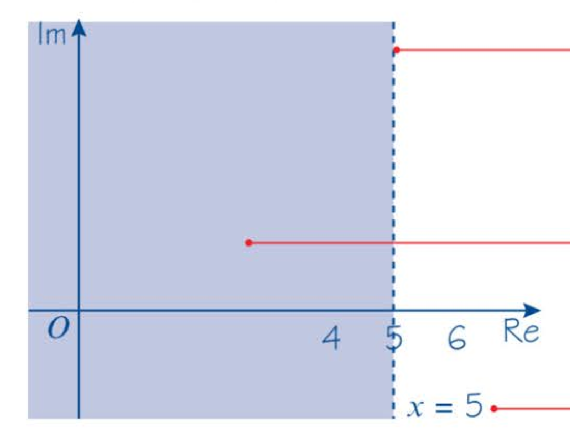


$|z - 4 - 2i| = 2$ represents a circle centre (4, 2), radius 2.

$|z - 4 - 2i| < 2$ represents the region on the inside of this circle.

$|z - 4 - 2i| \leqslant 2$ represents the boundary and the region inside of this circle.

ii $|z - 4| < |z - 6|$

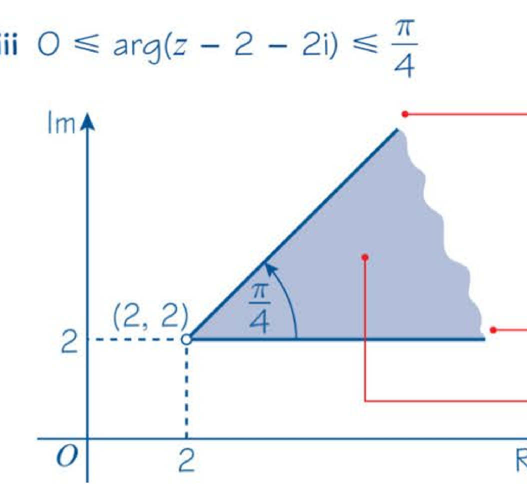


$|z - 4| = |z - 6|$ is represented by the line $x = 5$. This line is the perpendicular bisector of the line segment joining (4, 0) to (6, 0).

$|z - 4| < |z - 6|$ represents the region $x < 5$. All points in this region are closer to (4, 0) than to (6, 0).

Note this region does not include the line $x = 5$. So $x = 5$ is represented by a dashed line.

iii $0 \leqslant \arg(z - 2 - 2i) \leqslant \frac{\pi}{4}$



$\arg(z - 2 - 2i) = \frac{\pi}{4}$ is the half-line from the point (2, 2) at angle $\frac{\pi}{4}$ to the horizontal.

$\arg(z - 2 - 2i) = 0$ is the other half-line shown from the point (2, 2).

$0 \leqslant \arg(z - 2 - 2i) \leqslant \frac{\pi}{4}$ is represented by the region in between and including these two half-lines.