Summary of key points

For example $\begin{pmatrix}2 & 1\\4 & 0\end{pmatrix}$ is a 2 × 2 matrix and $\begin{pmatrix}1 & 4 & -1 & 1\\2 & 3 & 0 & 2\end{pmatrix}$ is a 2 × 4 matrix. Generally, you can refer to a matrix as $n \times m$ where n is the number of rows and m is the number of columns.

1 A **square matrix** is one where the numbers of rows and columns are the same.

2 A zero matrix is one in which all of the numbers are zero. The zero matrix is denoted by 0.

3 An identity matrix is a square matrix in which the numbers in the leading diagonal (starting top left) are 1 and all the rest are 0. Identity matrices are denoted by $\mathbf{I}_k$ where k describes the size. The 3 × 3 identity matrix is $\mathbf{I}_3 = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$

4 To add or subtract matrices, you add or subtract the corresponding elements in each matrix. You can only add or subtract matrices that are the same size.

Notation A **scalar** is a number rather than a matrix. In questions on matrices, scalars will be represented by non-bold letters and numbers.

5 To multiply a matrix by a scalar, you multiply every element in the matrix by that scalar.

6
- Matrices can be multiplied together if the number of columns in the first matrix is equal to the number of rows in the second matrix. In this case the first is said to be multiplicatively conformable with the second.
- To find the product of two multiplicatively conformable matrices, you multiply the elements in each row in the left-hand matrix by the corresponding elements in each column in the right-hand matrix, then add the results together.

Example 6

Given that $\mathbf{A} = \begin{pmatrix}-1 & 0\\2 & 3\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix}4 & 1\\0 & -2\end{pmatrix}$, find:

a **AB** **b** **BA**

a **A** is a 2 × 2 matrix and **B** is a 2 × 2 matrix so they can be multiplied and the product will be a 2 × 2 matrix.

$\mathbf{AB} = \begin{pmatrix}-1 & 0\\2 & 3\end{pmatrix}\begin{pmatrix}4 & 1\\0 & -2\end{pmatrix} = \begin{pmatrix}a & b\\c & d\end{pmatrix}$

$a = (-1) \times 4 + 0 \times 0 = -4$

$b = (-1) \times 1 + 0 \times (-2) = -1$

$c = 2 \times 4 + 3 \times 0 = 8$

$d = 2 \times 1 + 3 \times (-2) = -4$

So $\mathbf{AB} = \begin{pmatrix}-4 & -1\\8 & -4\end{pmatrix}$

b **BA** will also be a 2 × 2 matrix.

$\begin{pmatrix}4 & 1\\0 & -2\end{pmatrix}\begin{pmatrix}-1 & 0\\2 & 3\end{pmatrix} = \begin{pmatrix}-2 & 3\\-4 & -6\end{pmatrix}$

Example 8

$\mathbf{A} = (1 \quad -1 \quad 2)$, $\mathbf{B} = (3 \quad -2)$ and $\mathbf{C} = \begin{pmatrix}4\\5\end{pmatrix}$. Find **BCA**.

$\mathbf{BC} = (3 \quad -2)\begin{pmatrix}4\\5\end{pmatrix} = (2)$

$(\mathbf{BC})\mathbf{A} = (2)(1 \quad -1 \quad 2) = (2 \quad -2 \quad 4)$

This product could have been calculated by first working out **CA** and then multiplying **B** by this product. In general, matrix multiplication is **associative** (meaning that the bracketing makes no difference provided the order stays the same), so (**BC**)**A** = **B**(**CA**).

Watch out Although you can multiply matrices using a calculator, you need to know how the process works so that you can deal with matrices containing unknowns.

7 For a 2 × 2 matrix $\mathbf{M} = \begin{pmatrix}a & b\\c & d\end{pmatrix}$, the determinant of **M** is $ad - bc$.

The determinant is a scalar value

8
- If det **M** = 0 then **M** is a **singular** matrix.
- If det **M** ≠ 0 then **M** is a **non-singular** matrix.

Links Singular matrices do not have an **inverse**.

Notation You can write the determinant of **M** as det **M**, $|\mathbf{M}|$ or $\begin{vmatrix}a & b\\c & d\end{vmatrix}$. It is also sometimes written as Δ.

9 You find the determinant of a 3 × 3 matrix by reducing the 3 × 3 determinant to 2 × 2 determinants using the formula:

$$\begin{vmatrix}a & b & c\\d & e & f\\g & h & i\end{vmatrix} = a\begin{vmatrix}e & f\\h & i\end{vmatrix} - b\begin{vmatrix}d & f\\g & i\end{vmatrix} + c\begin{vmatrix}d & e\\g & h\end{vmatrix}$$

Watch out There is a minus sign in front of the second term.

10 The **minor** of an element in a 3 × 3 matrix is the determinant of the 2 × 2 matrix that remains after the row and column containing that element have been crossed out.

11 The **inverse** of a matrix **M** is the matrix $\mathbf{M}^{-1}$ such that $\mathbf{MM}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$.

12 If $\mathbf{M} = \begin{pmatrix}a & b\\c & d\end{pmatrix}$, then $\mathbf{M}^{-1} = \frac{1}{\det \mathbf{M}}\begin{pmatrix}d & -b\\-c & a\end{pmatrix}$.

You can find the inverse of a matrix using your calculator.

Note If det **M** = 0, you will not be able to find the inverse matrix, since $\frac{1}{\det \mathbf{M}}$ is undefined.

13 If **A** and **B** are non-singular matrices, then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Example 15

P and **Q** are non-singular matrices. Prove that $(\mathbf{PQ})^{-1} = \mathbf{Q}^{-1}\mathbf{P}^{-1}$.

Let $\mathbf{C} = (\mathbf{PQ})^{-1}$ then $(\mathbf{PQ})\mathbf{C} = \mathbf{I}$.

$\mathbf{P}^{-1}\mathbf{PQC} = \mathbf{P}^{-1}\mathbf{I}$

$(\mathbf{P}^{-1}\mathbf{P})\mathbf{QC} = \mathbf{P}^{-1}\mathbf{I}$

So $\mathbf{QC} = \mathbf{P}^{-1}$

$\mathbf{Q}^{-1}\mathbf{QC} = \mathbf{Q}^{-1}\mathbf{P}^{-1}$

$\mathbf{IC} = \mathbf{Q}^{-1}\mathbf{P}^{-1}$

$\mathbf{C} = \mathbf{Q}^{-1}\mathbf{P}^{-1}$

So $(\mathbf{PQ})^{-1} = \mathbf{Q}^{-1}\mathbf{P}^{-1}$ as required.

Example 16

A and **B** are non-singular 2 × 2 matrices such that **BAB** = **I**.

a Prove that $\mathbf{A} = \mathbf{B}^{-1}\mathbf{B}^{-1}$.

a $\mathbf{BAB} = \mathbf{I}$

$\mathbf{B}^{-1}\mathbf{BAB} = \mathbf{B}^{-1}\mathbf{I}$

$(\mathbf{B}^{-1}\mathbf{B})\mathbf{AB} = \mathbf{B}^{-1}\mathbf{I}$

$\mathbf{AB} = \mathbf{B}^{-1}$

$\mathbf{AB}\,\mathbf{B}^{-1} = \mathbf{B}^{-1}\,\mathbf{B}^{-1}$

$\mathbf{AI} = \mathbf{B}^{-1}\,\mathbf{B}^{-1}$

And hence $\mathbf{A} = \mathbf{B}^{-1}\,\mathbf{B}^{-1}$ as required.

14 The **transpose** of a matrix is found by interchanging the rows and the columns.

For example, if $\mathbf{A} = \begin{pmatrix}1 & 4\\2 & 3\end{pmatrix}$, $\mathbf{A}^{\mathrm{T}} = \begin{pmatrix}1 & 2\\4 & 3\end{pmatrix}$.

15 Finding the inverse of a 3 × 3 matrix **A** usually consists of the following 5 steps.

Step 1 Find the determinant of **A**, det **A**.

Step 2 Form the matrix of the minors of **A**, **M**.
In forming the matrix **M**, each of the nine elements of the matrix **A** is replaced by its minor.

Step 3 From the matrix of minors, form the matrix of **cofactors**, **C**, by changing the signs of some elements of the matrix of minors according to the **rule of alternating signs** illustrated by the pattern

$$\begin{pmatrix}+ & - & +\\- & + & -\\+ & - & +\end{pmatrix}$$

You leave the elements of the matrix of minors corresponding to the + signs in this pattern unchanged. You change the signs of the elements corresponding to the – signs.

Step 4 Write down the transpose, $\mathbf{C}^{\mathrm{T}}$, of the matrix of cofactors.

Step 5 The inverse of the matrix **A** is given by the formula

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\mathbf{C}^{\mathrm{T}}$$

Example 18

The matrix $\mathbf{A} = \begin{pmatrix}3 & 2 & -2\\-2 & k & 0\\-1 & -3 & 3\end{pmatrix}$, $k \neq 0$. Find $\mathbf{A}^{-1}$.

Step 1

$\det \mathbf{A} = 3\begin{vmatrix}k & 0\\-3 & 3\end{vmatrix} - 2\begin{vmatrix}-2 & 0\\-1 & 3\end{vmatrix} + (-2)\begin{vmatrix}-2 & k\\-1 & -3\end{vmatrix}$

$= 3(3k - 0) - 2(-6 - 0) - 2(6 + k)$

$= 9k + 12 - 12 - 2k = 7k$

Watch out Make sure you understand the steps needed to find the inverse of a 3 × 3 matrix. You won't be able to use your calculator if the matrix contains unknowns.

As you are given that $k \neq 0$, the matrix is non-singular and the inverse can be found.

Step 1

$\det \mathbf{A} = 1\begin{vmatrix}4 & 1\\-1 & 0\end{vmatrix} - 3\begin{vmatrix}0 & 1\\2 & 0\end{vmatrix} + 1\begin{vmatrix}0 & 4\\2 & -1\end{vmatrix}$

$= 1(0 + 1) - 3(0 - 2) + 1(0 - 8)$

$= 1 + 6 - 8 = -1$

The first step of finding the inverse of a matrix is to evaluate its determinant.

Step 2

$\mathbf{M} = \begin{pmatrix}\begin{vmatrix}4 & 1\\-1 & 0\end{vmatrix} & \begin{vmatrix}0 & 1\\2 & 0\end{vmatrix} & \begin{vmatrix}0 & 4\\2 & -1\end{vmatrix}\\ \begin{vmatrix}3 & 1\\-1 & 0\end{vmatrix} & \begin{vmatrix}1 & 1\\2 & 0\end{vmatrix} & \begin{vmatrix}1 & 3\\2 & -1\end{vmatrix}\\ \begin{vmatrix}3 & 1\\4 & 1\end{vmatrix} & \begin{vmatrix}1 & 1\\0 & 1\end{vmatrix} & \begin{vmatrix}1 & 3\\0 & 4\end{vmatrix}\end{pmatrix}$

$= \begin{pmatrix}1 & -2 & -8\\1 & -2 & -7\\-1 & 1 & 4\end{pmatrix}$

The second step is to form the matrix of minors. The minor of an element is found by deleting the row and the column in which the element lies, then finding the determinant of the resulting 2 × 2 matrix.

For example, to find the minor of 4 in $\begin{pmatrix}1 & 3 & 1\\0 & 4 & 1\\2 & -1 & 0\end{pmatrix}$, delete the row and column containing 4, $\begin{pmatrix}1 & 3 & 1\\0 & 4 & 1\\2 & -1 & 0\end{pmatrix}$. The minor is the determinant of the elements left, $\begin{vmatrix}1 & 1\\2 & 0\end{vmatrix} = -2$.

Step 3

$\mathbf{C} = \begin{pmatrix}1 & 2 & -8\\-1 & -2 & 7\\-1 & -1 & 4\end{pmatrix}$

You find the matrix of cofactors by adjusting the signs of the minors using the pattern $\begin{pmatrix}+ & - & +\\- & + & -\\+ & - & +\end{pmatrix}$. Here you leave the elements $\begin{pmatrix}1 & & -8\\ & -2 & \\-1 & & 4\end{pmatrix}$ unchanged but change the signs of $\begin{pmatrix} & -2 & \\1 & & -7\\ & 1 & \end{pmatrix}$.

Step 4

$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix}1 & -1 & -1\\2 & -2 & -1\\-8 & 7 & 4\end{pmatrix}$

Step 5

$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\mathbf{C}^{\mathrm{T}} = \frac{1}{-1}\begin{pmatrix}1 & -1 & -1\\2 & -2 & -1\\-8 & 7 & 4\end{pmatrix}$

$= \begin{pmatrix}-1 & 1 & 1\\-2 & 2 & 1\\8 & -7 & -4\end{pmatrix}$

You divide each term of the transpose of the matrix of cofactors, $\mathbf{C}^{\mathrm{T}}$, by the determinant of **A**, –1.

Problem-solving Proving $\mathbf{A} = \mathbf{A}^{-1}$ is equivalent to proving $\mathbf{A}^2 = \mathbf{I}$. You still need to add working to show that $\mathbf{A}^2 = \mathbf{I}$ implies that $\mathbf{A} = \mathbf{A}^{-1}$.

$\mathbf{AA} = \mathbf{I}$

$\mathbf{A}^{-1}\mathbf{AA} = \mathbf{A}^{-1}$

$\mathbf{A} = \mathbf{A}^{-1}$ as required

Notation If $\mathbf{A}^{-1} = \mathbf{A}$, then the matrix **A** is said to be **self-inverse**.

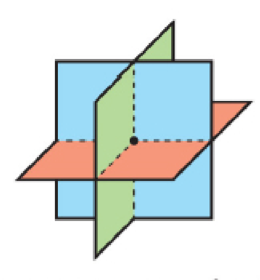
16 If $\mathbf{A}\begin{pmatrix}x\\y\\z\end{pmatrix} = \mathbf{v}$ then $\begin{pmatrix}x\\y\\z\end{pmatrix} = \mathbf{A}^{-1}\mathbf{v}$.

17 A system of linear equations is **consistent** if there is at least one set of values that satisfies all the equations simultaneously. Otherwise, it is **inconsistent**.

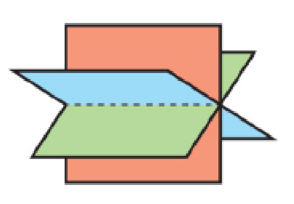
If the matrix corresponding to a set of linear equations is non-singular, then the system has one unique solution and is consistent. However, if the matrix is singular, there are two possibilities: either the system is consistent and has infinitely many solutions, or it is inconsistent and has no solutions.

You can visualise the different situations by considering the points of intersection of the planes corresponding to each equation. Here are some of the different possible configurations:

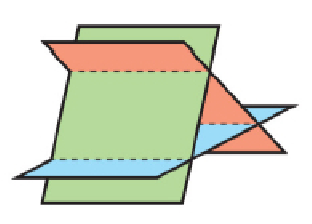
Links An equation in the form $ax + by + cz = d$ is the equation of a plane in three dimensions.



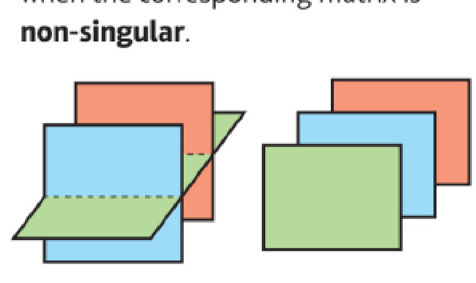
The planes meet at a **point**. The system of equations is **consistent** and has **one solution** represented by this point. This is the only case when the corresponding matrix is **non-singular**.



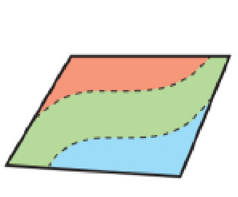
The planes form a **sheaf**. The system of equations is **consistent** and has **infinitely many solutions** represented by the line of intersection of the three planes.



The planes form a **prism**. The system of equations is **inconsistent** and has **no solutions**.



Two or more of the planes are parallel and non-identical. The system of equations is **inconsistent** and has **no solutions**.



All three equations represent the same plane. In this case the system of equations is **consistent** and has **infinitely many solutions**.

Hint If one row of the corresponding matrix is a **linear multiple** of another row, then these two rows will represent parallel planes.