## Summary of key points

The new point is called the **image**.

1 • Linear transformations always map the origin onto itself.
• Any linear transformation can be represented by a matrix.

A **linear transformation** has the special properties that the transformation only involves linear terms in $x$ and $y$.

**Watch out** $S$ represents a translation, but this is not a linear transformation since you can't write $x + 4$ in the form $ax + by$.

Points which are mapped onto themselves under the given transformation are called **invariant points**. Lines which map onto themselves are called **invariant lines**.

2 The linear transformation $T$: $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$ can be represented by the matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

since $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$.

Consider the unit square, and the effect that the transformation has on the unit vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. This will completely define the transformation.

**Problem-solving**

Each point on the mirror line is invariant, so this line is an invariant line. Any line perpendicular to the mirror line will also be an invariant line, although the points on the line will not, in general, be invariant points.

3 A reflection in the $y$-axis is represented by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Points on the $y$-axis are invariant points, and the lines $x = 0$ and $y = k$ for any value of $k$ are invariant lines.

4 A reflection in the $x$-axis is represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Points on the $x$-axis are invariant points, and the lines $y = 0$ and $x = k$ for any value of $k$ are invariant lines.

5 A reflection in the line $y = x$ is represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Points on the line $y = x$ are invariant points, and the lines $y = x$ and $y = -x + k$ for any value of $k$ are invariant lines.

6 A reflection in the line $y = -x$ is represented by the matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. Points on the line $y = -x$ are invariant points, and the lines $y = -x$ and $y = x + k$ for any value of $k$ are invariant lines.

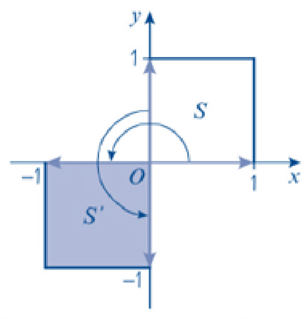
**Example 6**

The transformation $U$, represented by the $2 \times 2$ matrix **P**, is a rotation of 180° about the point (0, 0).

**a** Write down the matrix **P**.

**b** Show that the line $y = 3x$ is invariant under this transformation.

**a** The given rotation is shown in the diagram:




If you need to find the matrix that represents a given transformation, it can help to draw a sketch transforming the unit square. Remember the transformation is defined by its effect on the unit vectors.

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ maps to $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ maps to $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

Hence the matrix is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

**b** $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ 3x \end{pmatrix} = \begin{pmatrix} -x \\ -3x \end{pmatrix}$

Since $-3x = 3(-x)$, this point lies on the line $y = 3x$.

So points on the line $y = 3x$ are mapped to points on the line $y = 3x$.

Hence $y = 3x$ is an invariant line.

**Problem-solving**

Write a general point on the line $y = 3x$ as $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 3x \end{pmatrix}$. Apply the transformation to this point, then show that the image also lies on the line. Watch out: although the line is invariant, the only invariant **point** on the line is the origin.

7 The matrix representing a rotation through angle $\theta$ anticlockwise about the origin is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

The only invariant point is the origin (0, 0). For $\theta \neq 180°$, there are no invariant lines. For $\theta = 180°$, any line passing through the origin is an invariant line.

8 A transformation represented by the matrix $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ is a stretch of scale factor $a$ parallel to the $x$-axis and a stretch of scale factor $b$ parallel to the $y$-axis.

In the case where $a = b$, the transformation is an enlargement with scale factor $a$.

9 For any stretch of the above form, the $x$- and $y$-axes are invariant lines and the origin is an invariant point.

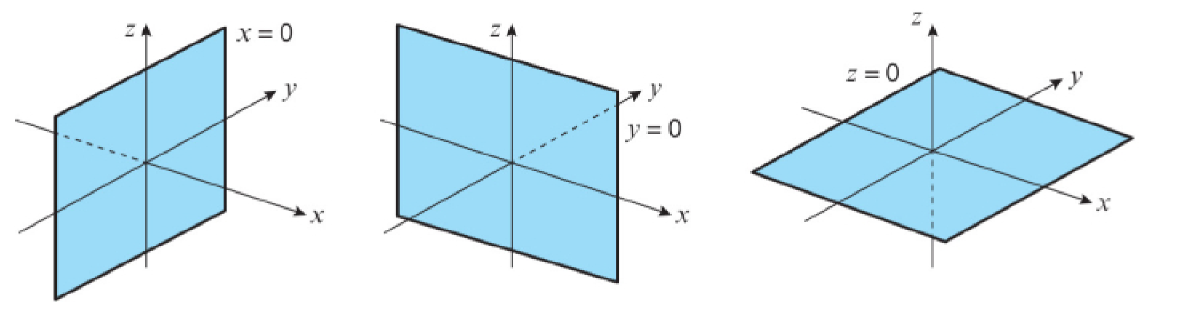
10 For a stretch parallel to the $x$-axis only, points on the $y$-axis are invariant points, and any line parallel to the $x$-axis is an invariant line.

11 For a stretch parallel to the $y$-axis only, points on the $x$-axis are invariant points, and any line parallel to the $y$-axis is an invariant line.

12 For a linear transformation represented by matrix **M**, det **M** represents the scale factor for the change in area. This is sometimes called the **area scale factor**.

**Watch out** If the determinant of the matrix **M** is negative, the shape has been reflected.

13 The matrix **PQ** represents the transformation $Q$, with matrix **Q**, followed by the transformation $P$, with matrix **P**.




14 A reflection in the plane $x = 0$ is represented by the matrix $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

15 A reflection in the plane $y = 0$ is represented by the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

16 A reflection in the plane $z = 0$ is represented by the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

17 A rotation, angle $\theta$, anticlockwise about the $x$-axis is represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

18 A rotation, angle $\theta$, anticlockwise about the $y$-axis is represented by the matrix

$$\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

19 A rotation, angle $\theta$, anticlockwise about the $z$-axis is represented by the matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

20 The transformation described by the matrix $\mathbf{A}^{-1}$ has the effect of reversing the transformation described by the matrix **A**.

You can solve two equations in two unknowns quickly using your calculator.