Summary of key points

1 If α and β are roots of the equation $ax^2 + bx + c = 0$, then:

- $\alpha + \beta = -\frac{b}{a}$
- $\alpha\beta = \frac{c}{a}$

Links If the roots of a quadratic equation with real coefficients are complex, then they occur as a conjugate pair.

2 If α, β and γ are roots of the equation $ax^3 + bx^2 + cx + d = 0$, then:

- $\alpha + \beta + \gamma = \Sigma\alpha = -\frac{b}{a}$
- $\alpha\beta + \beta\gamma + \gamma\alpha = \Sigma\alpha\beta = \frac{c}{a}$
- $\alpha\beta\gamma = -\frac{d}{a}$

Links If a cubic equation with real coefficients has two complex roots, then they will occur as a conjugate pair.

3 If α, β, γ and δ are roots of the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, then:

- $\alpha + \beta + \gamma + \delta = \Sigma\alpha = -\frac{b}{a}$
- $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \Sigma\alpha\beta = \frac{c}{a}$
- $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = \Sigma\alpha\beta\gamma = -\frac{d}{a}$
- $\alpha\beta\gamma\delta = \frac{e}{a}$

Notation You can use the following abbreviations for these results in your working:
$\Sigma\alpha = -\frac{b}{a}$ $\quad\Sigma\alpha\beta = \frac{c}{a}$ $\quad\Sigma\alpha\beta\gamma = -\frac{d}{a}$

4 The rules for **reciprocals**:

- Quadratic: $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta}$
- Cubic: $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma}$
- Quartic: $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = \frac{\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta}{\alpha\beta\gamma\delta}$

Note If you learn these you can use them without proof in your exam.

5 The rules for **products of powers**:

- Quadratic: $\alpha^n \times \beta^n = (\alpha\beta)^n$
- Cubic: $\alpha^n \times \beta^n \times \gamma^n = (\alpha\beta\gamma)^n$
- Quartic: $\alpha^n \times \beta^n \times \gamma^n \times \delta^n = (\alpha\beta\gamma\delta)^n$

6 The rules for **sums of squares**:

- Quadratic: $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
- Cubic: $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$
- Quartic: $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$

7 The rules for **sums of cubes**:

- Quadratic: $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$
- Cubic: $\alpha^3 + \beta^3 + \gamma^3 = (\alpha + \beta + \gamma)^3 - 3(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) + 3\alpha\beta\gamma$

Note The result for the sum of cubes for a quartic equation is not required.

Example 9

The quartic equation $x^4 - 3x^3 + 15x + 1 = 0$ has roots α, β, γ and δ. Find the equation with roots $(2\alpha + 1)$, $(2\beta + 1)$, $(2\gamma + 1)$ and $(2\delta + 1)$.

Method 1

$\alpha + \beta + \gamma + \delta = 3$
$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = 0$
$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -15$
$\alpha\beta\gamma\delta = 1$

Sum of roots:
$(2\alpha + 1) + (2\beta + 1) + (2\gamma + 1) + (2\delta + 1)$
$= 2(a + \beta + \gamma + \delta) + 4$
$= 10$

Pair sum:
$4(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta) + 6(\alpha + \beta + \gamma + \delta) + 6$
$= 4 \times 0 + 6 \times 3 + 6 = 24$

Triple sum:
$8(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta) + 8(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$
$+ 6(\alpha + \beta + \gamma + \delta) + 4 = -98$

Product:
$16\alpha\beta\gamma\delta + 8(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta) + 4(\alpha\beta + \alpha\gamma$
$+ \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta) + 2(\alpha + \beta + \gamma + \delta) + 1$
$= -97$

Hence the new equation is
$w^4 - 10w^3 + 24w^2 + 98w - 97 = 0$

Method 2

Let $w = 2x + 1$ hence $x = \frac{w - 1}{2}$

Substituting: $\left(\frac{w-1}{2}\right)^4 - 3\left(\frac{w-1}{2}\right)^3 + 15\left(\frac{w-1}{2}\right) + 1 = 0$

$(w - 1)^4 - 6(w - 1)^3 + 120(w - 1) + 16 = 0$
$w^4 - 4w^3 + 6w^2 - 4w + 1 - 6(w^3 - 3w^2 + 3w - 1)$
$+ 120w - 120 + 16 = 0$
$w^4 - 10w^3 + 24w^2 + 98w - 97 = 0$

Problem-solving In many cases, it is quicker to use a substitution.