Summary of key points

1 If the displacement, s, is expressed as a function of t, then the velocity, v, can be expressed as $v = \frac{ds}{dt}$

2 If the velocity, v, is expressed as a function of t, then the acceleration, a, can be expressed as $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$

3

$$\text{Differentiate} \downarrow \qquad \begin{aligned} \text{displacement} &= s = \int v\,dt \\ \frac{ds}{dt} = \text{velocity} &= v = \int a\,dt \\ \frac{dv}{dt} = \frac{d^2s}{dt^2} = \text{acceleration} &= a \end{aligned} \qquad \uparrow \text{Integrate}$$

Had a look ☐ Nearly there ☐ Nailed it! ☐

Variable acceleration 1

You can write displacement, velocity and acceleration as **functions of time.** This allows you to model the motion of an object which is moving in a straight line with variable acceleration.

Worked example

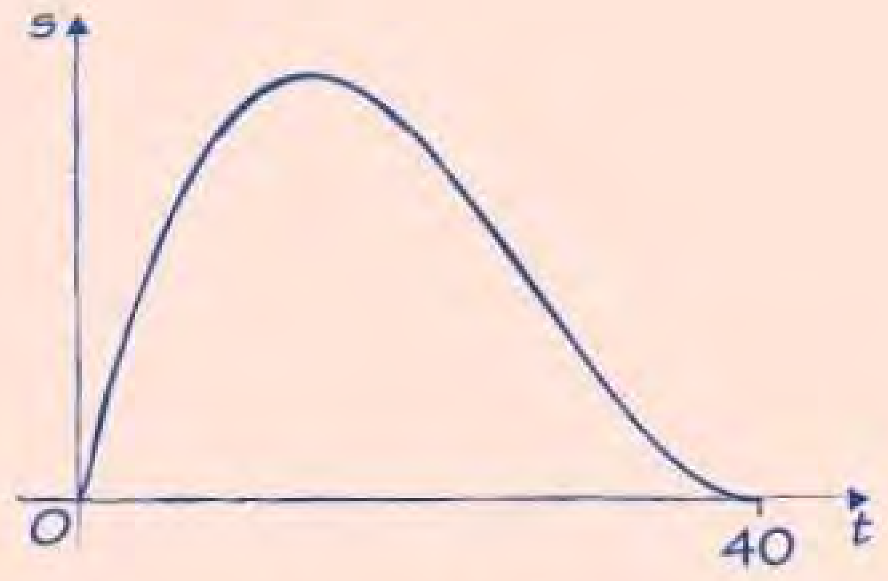
A pipe-inspection robot travels along a straight section of sewer pipe. At time t minutes the distance, s metres, of the robot from its starting position is given by

$$s = \frac{t^3 - 80t^2 + 1600t}{50},\ 0 \leqslant t \leqslant 40$$

(a) Sketch a distance–time graph for the motion of the robot. **(3 marks)**

$s = \frac{1}{50}t(t^2 - 80t + 1600)$

$= \frac{1}{50}t(t - 40)^2$



(b) Find the maximum distance of the robot from its starting position. **(6 marks)**

$\frac{ds}{dt} = \frac{1}{50}(3t^2 - 160t + 1600)$

$0 = \frac{1}{50}(3t - 40)(t - 40)$

$t = \frac{40}{3}$ or $\cancel{t = 40}$

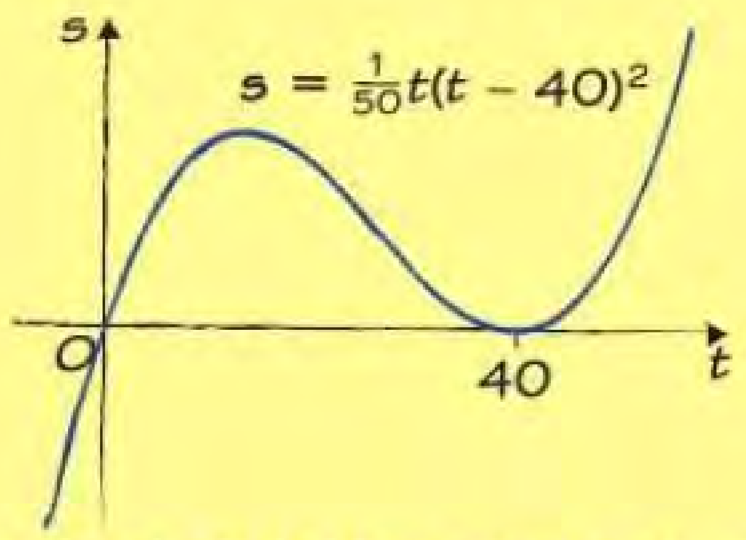
When $t = \frac{40}{3}$,

$s = \frac{1}{50}\left(\left(\frac{40}{3}\right)^3 - 80\left(\frac{40}{3}\right)^2 + 1600\left(\frac{40}{3}\right)\right)$

$= 189.62\ldots$

The maximum distance of the robot from its starting point is 190 m (3 s.f.)

You need to know more information about how the function behaves before you can sketch the distance–time graph. You can take out a factor of t and then factorise the quadratic factor. The graph of $s = \frac{1}{50}t(t - 40)^2$ is a cubic graph with a positive coefficient of t^3. It crosses the t-axis at $t = 0$ and touches it at $t = 40$:



There is more about sketching graphs of cubic functions on page 12.

The model provided is only valid for $0 \leqslant t \leqslant 40$. Make sure you only sketch the section of the graph for these values of t.

From your sketch you can see that $t = 40$ represents a minimum value, so reject that solution.

Maxima and minima

You can use differentiation to find any maximum and minimum values of a function of time. To find the maximum value of $s = f(t)$, differentiate and set $f'(t) = 0$. Solve to find t and then substitute this value back into $f(t)$.

The derivative of the displacement function represents the **velocity**. There is more about this on the next page. There is more about differentiating and finding minima and maxima on pages 39 and 40.

Now try this

A particle travels along the x-axis such that its distance, x m, from the origin at time t seconds is given by:

$x = t^4 - 20t^3 + 96t^2,\ 0 \leqslant t \leqslant 8$

(a) Show that x is non-negative for all values in the given domain of t. Answer **(3 marks)**

(b) Find the maximum distance of the particle from the origin, and justify that it is a maximum. Answer **(8 marks)**

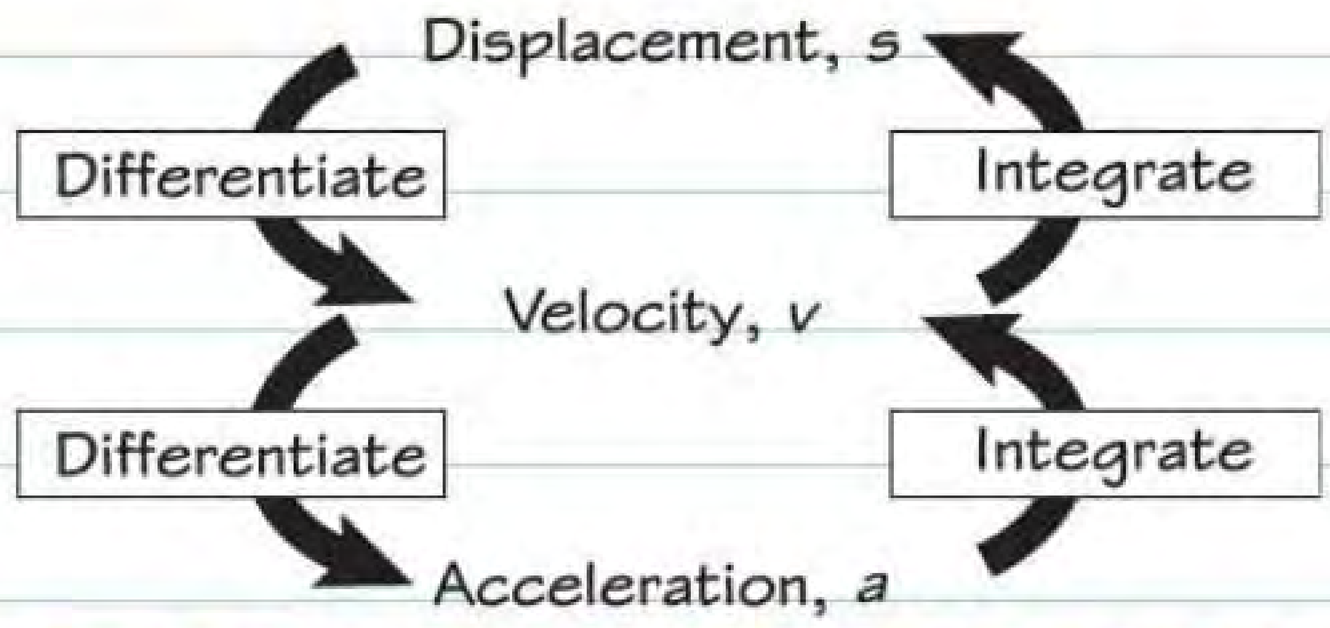
Have a look at page 12 for a reminder about how **quartic** functions behave.

Variable acceleration 2

Velocity is the rate of change of displacement, and acceleration is the rate of change of velocity.

$$v = \frac{ds}{dt} \text{ and } a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Integration is the **inverse** of differentiation, so you can integrate an expression for the velocity to find the displacement, or you can integrate an expression for the acceleration to find the velocity.



Worked example

A particle P moves along the x-axis, passing the origin at time $t = 0$, with velocity $17.6\ \text{m s}^{-1}$ in the direction of x increasing.

After a time t seconds, the acceleration of P in the direction of x increasing is $4 - 0.2t\ \text{m s}^{-2}$, $t \geqslant 0$.

Find

(a) the time at which P comes instantaneously to rest. **(6 marks)**

$v = \int (4 - 0.2t)\,dt$

$= 4t - 0.1t^2 + c$

When $t = 0$, $v = 17.6$

$17.6 = 4(0) - 0.1(0)^2 + c \Rightarrow c = 17.6$

So $v = 4t - 0.1t^2 + 17.6$

Set $v = 0$ and solve:

$-0.1t^2 + 4t + 17.6 = 0$

$t = \cancel{-4}$ or $t = 44$

P comes to rest after 44 seconds.

(b) the distance travelled by the particle in the first two seconds of its motion. **(4 marks)**

$s = \int_0^2 (4t - 0.1t^2 + 17.6)\,dt$

$= \left[2t^2 - \frac{1}{30}t^3 + 17.6t\right]_0^2$

$= \left(2(2)^2 - \frac{1}{30}(2)^3 + 17.6(2)\right) - 0$

$= 42.9\text{m (3 s.f.)}$

Completing the function

An indefinite integral always produces a function with a **constant of integration**. You need to find this constant in order to use the function to solve numerical problems. When you integrate to find velocity or displacement, you might need to use a given value to find the constant of integration. If this is the value when $t = 0$ it is called an **initial condition**.

There is more about finding functions by integrating on page 43.

Problem solved!

Follow these steps for part (a):

1. Integrate the expression for the acceleration of P to find an expression for the velocity.
2. Use the initial conditions to find the value of the constant of integration.
3. Set the velocity equal to 0 and solve. The model is only valid for $t \geqslant 0$ so you can reject the negative root of the equation.

You will need to use problem-solving skills throughout your exam – **be prepared!**

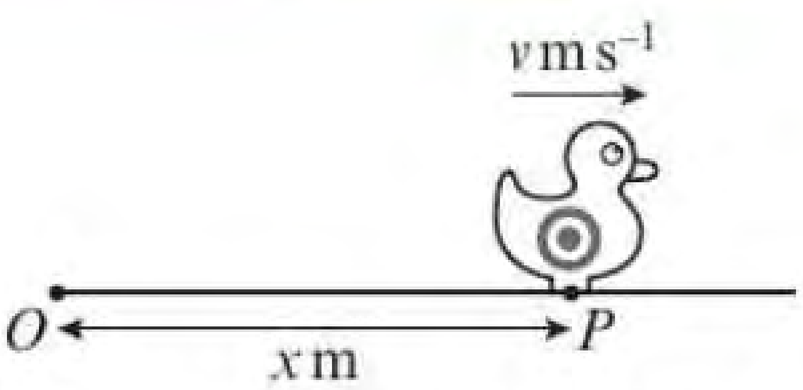
You can use **definite integration** to find the distance travelled between two given times. This is the same as the area under the velocity–time graph between $t = 0$ and $t = 2$.

Now try this

A target in a fairground shooting range moves along a straight line. At time t seconds, its distance from a fixed point O is x m and its velocity, $v\ \text{m s}^{-1}$, is given by: $v = 0.1\sqrt{t} - 0.2t$, $0 \leqslant t \leqslant 6$

When $t = 0$, $x = 4$.

(a) Find the acceleration of the target when $t = 4$. Answer **(4 marks)**



(b) Find an expression for x, giving your answer in the form $x = f(t)$ Answer **(4 marks)**

(c) Find the value of x when $t = 5$. **(1 mark)** Answer