Summary of key points

1 To solve a quadratic equation by factorising:

- Write the equation in the form $ax^2 + bx + c = 0$
- Factorise the left-hand side
- Set each factor equal to zero and solve to find the value(s) of x

2 The solutions of the equation $ax^2 + bx + c = 0$ where $a \neq 0$ are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3 $x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$

4 $ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$

5 The set of possible inputs for a function is called the **domain**.

The set of possible outputs of a function is called the **range**.

6 The **roots** of a function are the values of x for which $f(x) = 0$.

7 You can find the coordinates of a **turning point** of a quadratic graph by completing the square. If $f(x) = a(x + p)^2 + q$, the graph of $y = f(x)$ has a turning point at $(-p, q)$.

8 For the quadratic function $f(x) = ax^2 + bx + c = 0$, the expression $b^2 - 4ac$ is called the **discriminant**. The value of the discriminant shows how many roots $f(x)$ has:

- If $b^2 - 4ac > 0$ then a quadratic function has two distinct real roots.
- If $b^2 - 4ac = 0$ then a quadratic function has one repeated real root.
- If $b^2 - 4ac < 0$ then a quadratic function has no real roots.

9 Quadratics can be used to model real-life situations.

Quadratic equations

Quadratic equations can be written in the form $ax^2 + bx + c = 0$, where a, b and c are constants. The solutions of a quadratic equation are sometimes called the **roots** of the equation.

Solution by factorising

You can follow these steps to solve some quadratic equations.

1. **Rearrange** the equation into the form $ax^2 + bx + c = 0$
2. **Factorise** the left-hand side.
3. Set each factor **equal to zero** and solve to find two values of x.

The first solution is the value of x which makes the $(2x + 3)$ factor equal to 0.

Worked example

Solve $2(x + 1)^2 = 3x + 5$ **(4 marks)**

$2(x^2 + 2x + 1) = 3x + 5$

$2x^2 + 4x + 2 = 3x + 5$

$2x^2 + x - 3 = 0$

$(2x + 3)(x - 1) = 0$

$2x + 3 = 0$ or $x - 1 = 0$

$x = -\frac{3}{2}$ $x = 1$

Worked example

$x^2 + 6x - 2 = (x + a)^2 + b$, where a and b are constants.

(a) Find the values of a and b. **(3 marks)**

$x^2 + 6x - 2 = (x + 3)^2 - 3^2 - 2$

$= (x + 3)^2 - 9 - 2$

$= (x + 3)^2 - 11$

$a = 3$ and $b = -11$

(b) Hence, or otherwise, show that the roots of

$x^2 + 6x - 2 = 0$

are $c \pm \sqrt{11}$, where c is an integer to be found. **(2 marks)**

$(x + 3)^2 - 11 = 0$ $(+ 11)$

$(x + 3)^2 = 11$ $(\sqrt{\ })$

$x + 3 = \pm\sqrt{11}$ (-3)

$x = -3 \pm \sqrt{11}$

$c = -3$

Completing the square

You can write a quadratic expression in the form $(x + p)^2 + q$ using these two identities:

1. $x^2 + 2bx + c \equiv (x + b)^2 - b^2 + c$
2. $x^2 - 2bx + c \equiv (x - b)^2 - b^2 + c$

You can use this method to solve a quadratic equation without using a calculator.

Write the left-hand side in completed square form, and use inverse operations to solve the equation. Remember that any positive number has **two** square roots: one positive and one negative. You need to use the $\pm$ symbol when you take square roots of both sides of the equation.

Now try this

1 Solve the equation $2(x - 3)^2 + 3x = 14$ **(3 marks)**

Answer

2 (a) Show that $x^2 - 10x + 7$ can be written as $(x + p)^2 + q$, where p and q are integers to be found. **(2 marks)**

Answer

(b) Hence solve the equation $x^2 - 10x + 7 = 0$, giving your answer in the form $x = a \pm b\sqrt{2}$, where a and b are integers to be found. **(3 marks)**

Answer

$x^2 - 10x + 7 = (x - 5)^2 - 5^2 + 7$

You could also solve this by substituting $a = 1$, $b = -10$ and $c = 7$ into the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, then simplifying.

Had a look ☐ Nearly there ☐ Nailed it! ☐

Functions and roots

You need to be able to use function notation and to be able to solve quadratic equations in a **function of the unknown.** Here is an example:

You don't know how to solve a quartic equation like this, but if you write $u = x^2$ then $u^2 = (x^2)^2 = x^4$.

$$x^4 - 7x^2 + 12 = 0$$
$$u^2 - 7u + 12 = 0$$

Substitute $u = x^2$ to form a quadratic equation in u.

$$(u - 3)(u - 4) = 0$$
$$u = 3 \quad \text{or} \quad u = 4$$

Use the substitution to convert back to x. This equation has **four solutions.**

$$x^2 = 3 \qquad x^2 = 4$$
$$x = \pm\sqrt{3} \qquad x = \pm 2$$

You could also write
$(x^2)^2 - 7(x^2) + 12 = 0$
$(x^2 - 3)(x^2 - 4) = 0$

Worked example

Solve the equation $3x + \sqrt{x} - 2 = 0$ **(4 marks)**

Let $u = \sqrt{x}$

$3x + \sqrt{x} - 2 = 0$

$3u^2 + u - 2 = 0$

$(3u - 2)(u + 1) = 0$

$3u - 2 = 0$ or $u + 1 = 0$

$u = \frac{2}{3}$ or ~~$u = -1$~~

Using $u = \sqrt{x}$

$x = u^2$

$x = \frac{4}{9}$

Problem solved!

$x = (\sqrt{x})^2$ so you could write the equation as:

$$3(\sqrt{x})^2 + \sqrt{x} - 2 = 0$$

This is a **quadratic equation in** $\sqrt{x}$. The safest way to solve equations like this is to use the substitution $y = \sqrt{x}$:

$\sqrt{x}$ is the positive square root of x, so it can only take positive values. You need to ignore the solution $u = -1$.

You will need to use problem-solving skills throughout your exam – **be prepared!**

Domain

Functions will usually be defined for a given domain. This is the set of values that can be used as the input to the function. The domain of this function is all the positive **real numbers** ($\mathbb{R}$).

$$g(x) = 2x^2 - 5x - 3, x \in \mathbb{R}, x > 0$$

The **roots** of a function, $g(x)$, are the values of x for which $g(x) = 0$. You might need to consider the domain when finding the roots of a function.

Worked example

$g(x) = 2x^2 - 5x - 3, x \in \mathbb{R}, x > 0$

Show that $g(x)$ has exactly one root and find its value. **(3 marks)**

Set $g(x) = 0$ to find roots:

$2x^2 - 5x - 3 = 0$

$(2x + 1)(x - 3) = 0$

~~$x = -\frac{1}{2}$~~ or $x = 3$

The domain is $x > 0$, so $x = -\frac{1}{2}$ is not a root. The only root is $x = 3$.

Now try this

1 Solve

(a) $x^4 - 3x^2 - 4 = 0$ Answer **(4 marks)**

(b) $8x^6 + 7x^3 - 1 = 0$ Answer **(4 marks)**

(c) $x + 10 = 7\sqrt{x}$ Answer **(4 marks)**

2 Solve the equation $4\sqrt{x} + x = 3$, giving your answer in the form $a - b\sqrt{7}$, where a and b are integers to be found. **(5 marks)**

Answer

3 $f(x) = x^4 - 4x^2 - 5, x \in \mathbb{R}, x < 0$

Show that $f(x)$ has only one root and determine its exact value. **(4 marks)**

Answer

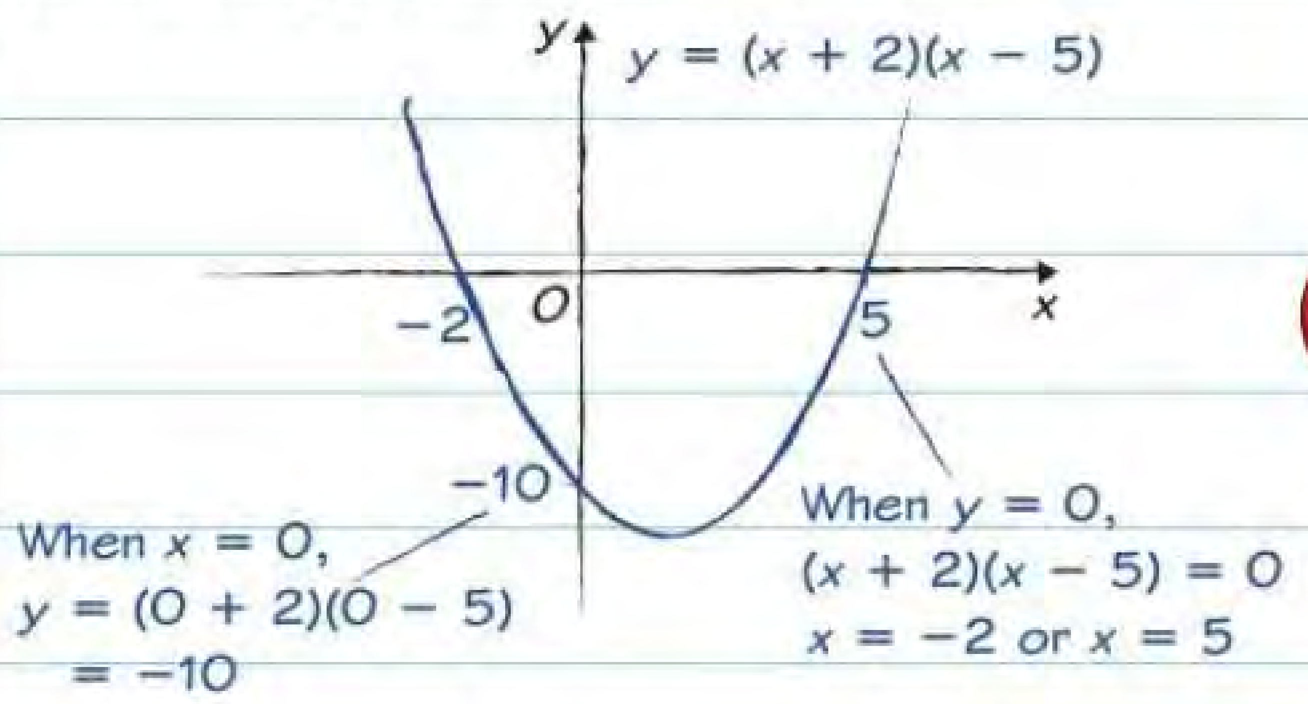
Look at the domain of the function carefully. $x^4 - 4x^2 - 5 = 0$ has **two real solutions** but only one of them is a root of $f(x)$.

Sketching quadratics

When you **sketch** a graph you need to show its key features. You don't need to use graph paper for a sketch, but you should still draw your axes and any straight lines using a **ruler**.

Factorised quadratics

On a sketch you usually show the points where the graph crosses the axes.

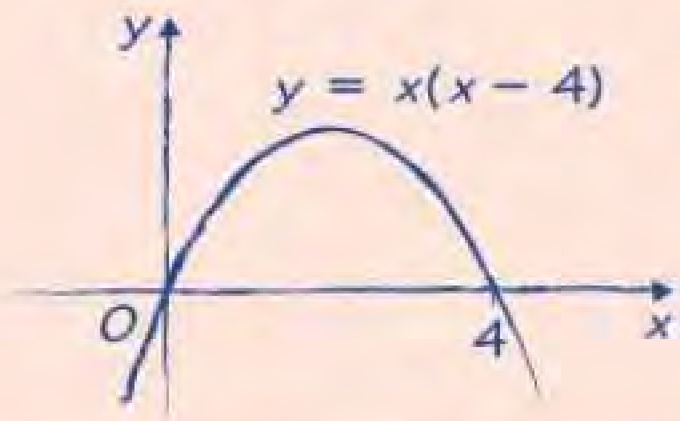


Negative coefficients

If the coefficient of x^2 is **negative** the graph will be an 'upside down' U-shape. You can check the coefficient of x^2 by multiplying out the brackets.

Worked example

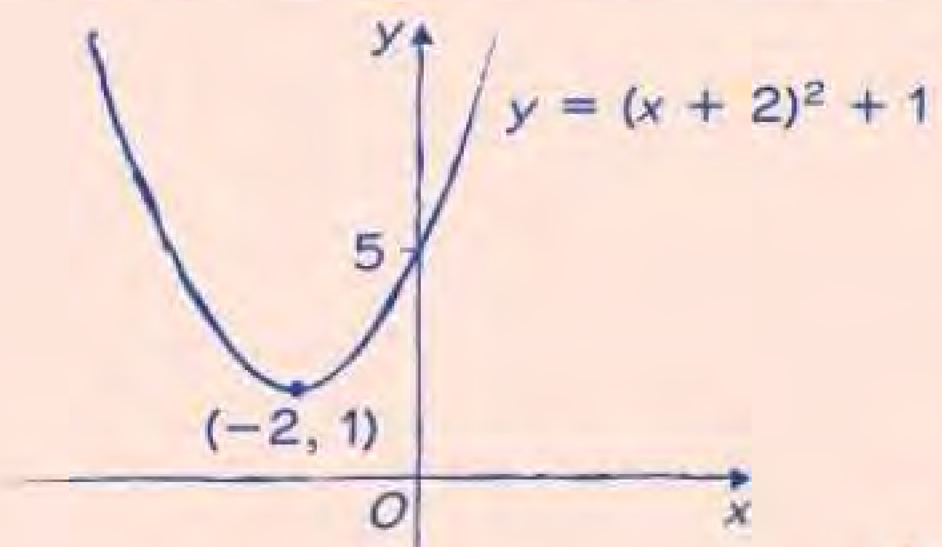
Sketch the curve with equation $y = x(4 - x)$ **(3 marks)**



$y = 4x - x^2$, so the coefficient of x^2 is **negative**.

Worked example

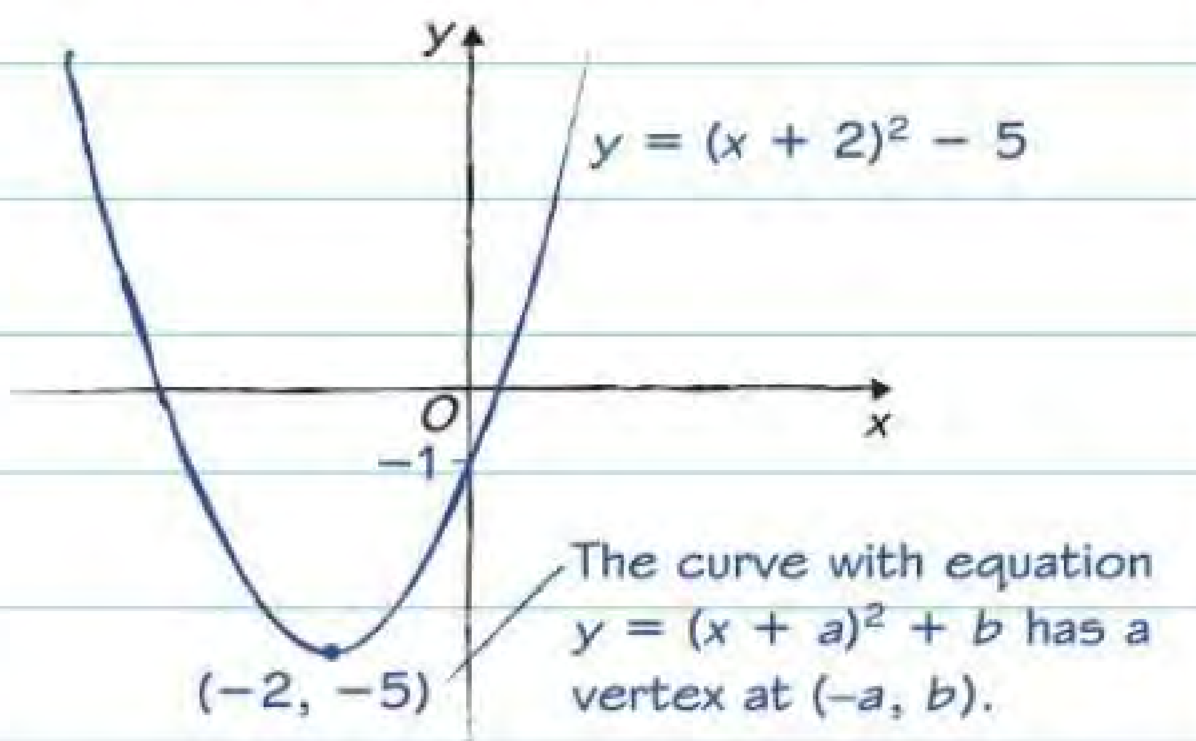
Sketch the curve with equation $y = (x + 2)^2 + 1$
Label the minimum point and any points where the curve crosses the coordinate axes. **(3 marks)**



To work out the point where the curve crosses the y-axis, substitute $x = 0$ into the equation.

Sketching $y = (x + a)^2 + b$

If a quadratic is written in **completed square** form then you can find the position of its **vertex** easily.



Now try this

1 On separate diagrams, sketch the graphs of

(a) $y = (x - 2)^2$ **(3 marks)**

(b) $y = (x - 2)^2 + k$, where k is a positive constant. **(2 marks)**

Show on each sketch the turning point and the coordinates of any points where the graph meets the axes.

Answer

Answer

2 $x^2 + 6x + 15 = (x + a)^2 + b$

Answer

(a) Find the values of a and b. **(2 marks)**

(b) Sketch the graph of $y = x^2 + 6x + 15$, labelling the minimum point and any points of intersection with the axes. **(3 marks)**

Answer

(c) Use your graph to explain why the equation $x^2 + 6x + 15 = 0$ has no real solutions. **(1 mark)**

Answer

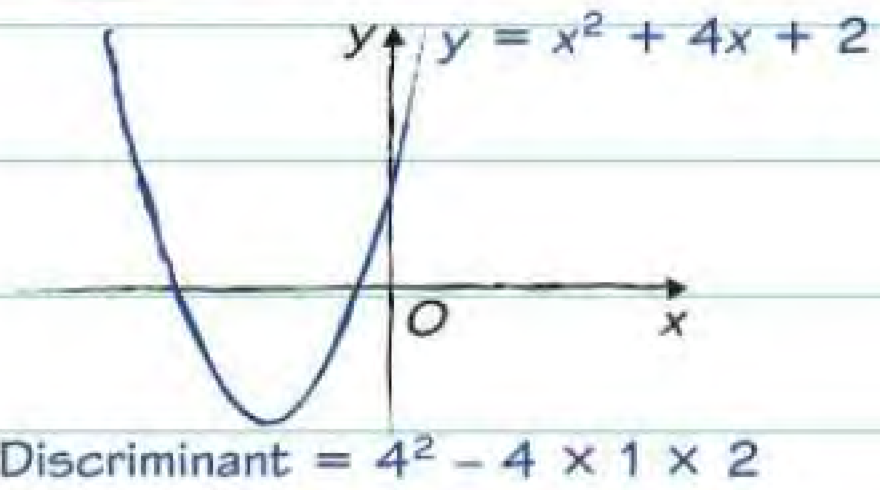
You can also use the discriminant to determine the number of real solutions. There is more about this on the next page.

Had a look ☐ Nearly there ☐ Nailed it! ☐

The discriminant

The discriminant of a quadratic expression $ax^2 + bx + c$ is the value $b^2 - 4ac$. You can use the discriminant to work out whether a quadratic equation has any **real roots** or **real solutions**. There are three possible conditions for the discriminant.

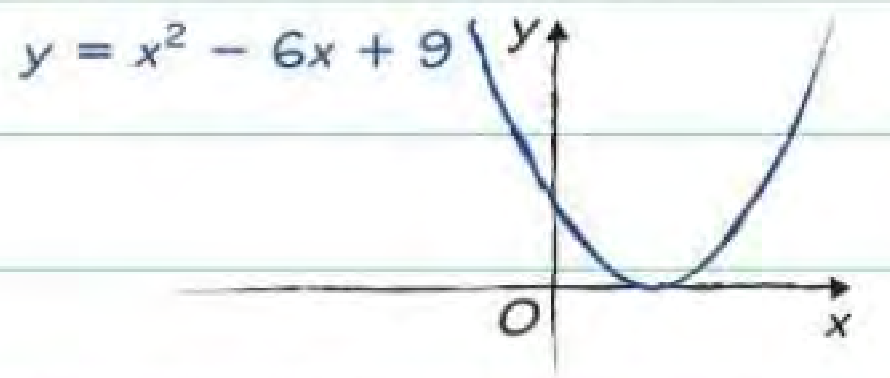
1 $b^2 - 4ac > 0$



Discriminant $= 4^2 - 4 \times 1 \times 2$
$= 8 > 0$

TWO DISTINCT REAL ROOTS

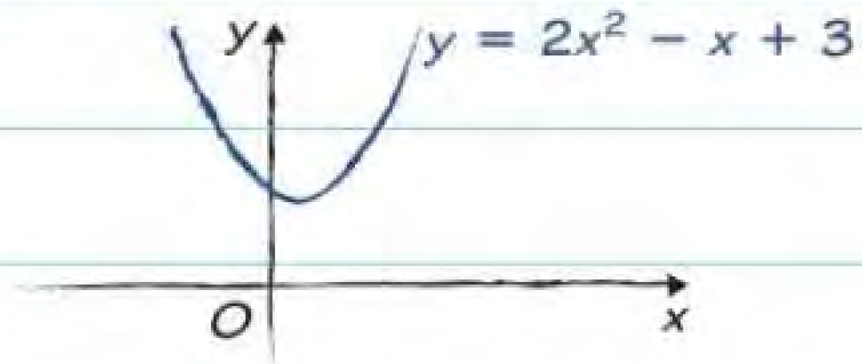
2 $b^2 - 4ac = 0$



Discriminant $= (-6)^2 - 4 \times 1 \times 9$
$= 0$

TWO EQUAL REAL ROOTS

3 $b^2 - 4ac < 0$



Discriminant $= (-1)^2 - 4 \times 2 \times 3$
$= -23 < 0$

NO REAL ROOTS

Worked example

The equation $x^2 + 4qx + 2q = 0$, where q is a non-zero constant, has equal roots.

Find the value of q. **(4 marks)**

$$b^2 - 4ac = 0$$
$$(4q)^2 - 4 \times 1 \times (2q) = 0$$
$$16q^2 - 8q = 0$$
$$q(16q - 8) = 0$$
$$\cancel{q = 0} \qquad 16q - 8 = 0 \text{ so } q = \tfrac{1}{2}$$

Follow these steps.

1. Work out the values of a, b and c: $a = 1$, $b = 4q$ and $c = 2q$.
2. Find an expression for the discriminant ($b^2 - 4ac$) in terms of q.
3. Set the discriminant equal to 0, because there are two **equal** roots.
4. Solve this **new** equation to work out two possible values for q.

You are told that q is non-zero, so the correct solution is $q = \frac{1}{2}$.

Problem solved!

The equation must be in the form $ax^2 + bx + c = 0$ before you work out the values of a, b and c. Always write down the condition for the discriminant that you are using, and use **brackets** when you substitute.

You will need to use problem-solving skills throughout your exam – **be prepared!**

Worked example

The equation $2x^2 - kx + 6 = k$ has no real solutions for x. Show that $k^2 + 8k - 48 < 0$ **(3 marks)**

$$2x^2 - kx + 6 - k = 0$$
$$b^2 - 4ac < 0$$
$$(-k)^2 - 4 \times 2 \times (6 - k) < 0$$
$$k^2 + 8k - 48 < 0$$

Now try this

1 Find the value of the discriminant of $3x^2 - 2x - 5$ **(1 mark)**

2 The equation $px^2 + 2x - 3 = 0$, where p is a constant, has equal roots. Find the value of p. **(3 marks)** Answer

Answer

The expression $(k + p)^2$ must always be greater than or equal to zero.

3 $f(x) = x^2 + (k + 5)x + 2k$, where k is a constant.

(a) Find the discriminant of $f(x)$ in terms of k. **(2 marks)** Answer

(b) Show that the discriminant can be written in the form $(k + p)^2 + q$, where p and q are integers to be found. **(2 marks)**

(c) Show that, for all values of k, the equation $f(x) = 0$ has distinct real roots. **(2 marks)** Answer

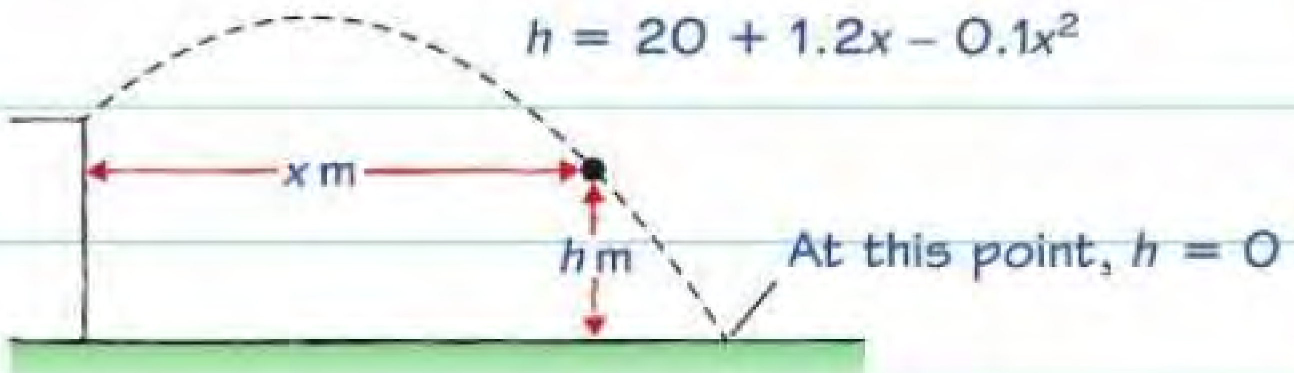
Answer

Modelling with quadratics

You can use quadratic functions and graphs to **model** real life situations.

Projectiles

An object moving freely under gravity is called a projectile. If you ignore air resistance, the paths of projectiles can be modelled as quadratic curves. This model describes the path of an object projected from the top of a 20 m building:



Worked example

A car manufacturer currently sells 50 000 units of a popular model each year, at a recommended retail price of £17 000 each. The manufacturer determines that for every £200 they increase the price, they will sell 400 fewer cars each year.

The number of cars sold each year, N, for a given price £P, is modelled as $N = a + bP$

(a) Determine the values of the constants a and b in this model. **(2 marks)**

$200b = -400$
$b = -2$

$50\,000 = a - 2(17\,000)$
$a = 84\,000$

The model is $N = 84\,000 - 2P$

The total revenue generated, £X, can be modelled as $X = P(a - bP)$, where a and b are the same constants.

(b) Rearrange X into the form $c - d(P - e)^2$, where c, d and e are constants to be found. **(3 marks)**

$X = P(84\,000 - 2P)$
$= -2P^2 + 84\,000P$
$= -2(P^2 - 42\,000P)$
$= -2[(P - 21\,000)^2 - 21\,000^2]$
$= 441\,000\,000 - 2(P - 21\,000)^2$

(c) State, with reasons, the amount of money the manufacturer should charge to maximise their revenue, and write down the maximum revenue. **(2 marks)**

The revenue will be maximised when $2(P - 21\,000)^2$ takes its minimum value (zero). This occurs when $P = 21\,000$, so the manufacturer should charge £21 000. This will generate a revenue of £441 million.

Problem solved!

The constant b in this model tells you how much N will change for every unit change in P. You are told that the change in N will be −400 when P increases by 200. Use this information to write an equation and solve it to find b. Then use your value of b together with the fact that $N = 50\,000$ when $P = 17\,000$ to find a.

You will need to use problem-solving skills throughout your exam – **be prepared!**

You need to complete the square for the expression $P(84\,000 - 2P)$. Start by multiplying out the brackets. Then take out a factor of −2 so the coefficient of P^2 is 1.

Your answer should refer to the **context of the question**. You should check your answer makes sense. £21 000 sounds about right for the price of a new car.

Now try this

A stone is thrown from a clifftop at an angle of 35° above the horizontal. The height, h metres, of the stone above the water when it is a horizontal distance d metres from the cliff is modelled by the formula:

$$h = 42 + 0.7d - 0.14d^2,\ d \geq 0$$

(a) Write down the height of the clifftop. **(1 mark)**

Answer

(b) Determine the horizontal distance travelled by the stone at the time it hits the water. **(3 marks)**

Answer

(c) By completing the square, or otherwise, determine the maximum height of the stone above the water during its flight. **(4 marks)**

Answer