Summary of key points

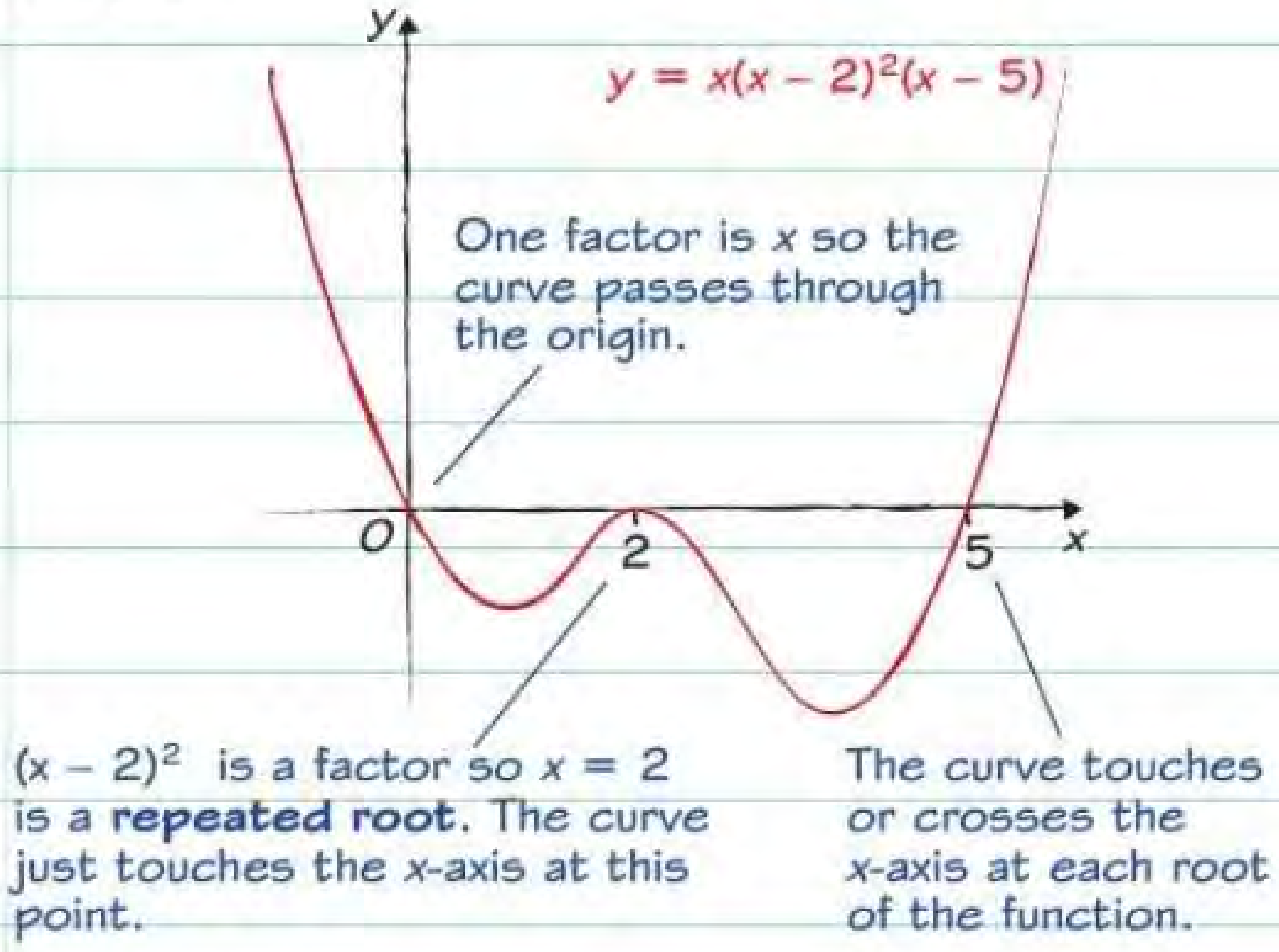
1. If p is a root of the function $f(x)$, then the graph of $y = f(x)$ touches or crosses the x-axis at the point $(p, 0)$.
2. The graphs of $y = \frac{k}{x}$ and $y = \frac{k}{x^2}$, where k is a real constant, have asymptotes at $x = 0$ and $y = 0$.
3. The x-coordinate(s) at the points of intersection of the curves with equations $y = f(x)$ and $y = g(x)$ are the solution(s) to the equation $f(x) = g(x)$.
4. The graph of $y = f(x) + a$ is a translation of the graph $y = f(x)$ by the vector $\begin{pmatrix} 0 \\ a \end{pmatrix}$.
5. The graph of $y = f(x + a)$ is a translation of the graph $y = f(x)$ by the vector $\begin{pmatrix} -a \\ 0 \end{pmatrix}$.
6. When you translate a function, any asymptotes are also translated.
7. The graph of $y = af(x)$ is a stretch of the graph $y = f(x)$ by a scale factor of a in the vertical direction.
8. The graph of $y = f(ax)$ is a stretch of the graph $y = f(x)$ by a scale factor of $\frac{1}{a}$ in the horizontal direction.
9. The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ in the x-axis.
10. The graph of $y = f(-x)$ is a reflection of the graph of $y = f(x)$ in the y-axis.

Cubic and quartic graphs

In a cubic function, the highest power of x is x^3. In a quartic function, it is x^4. You need to know the shapes of graphs of cubic and quartic functions and be able to sketch them.

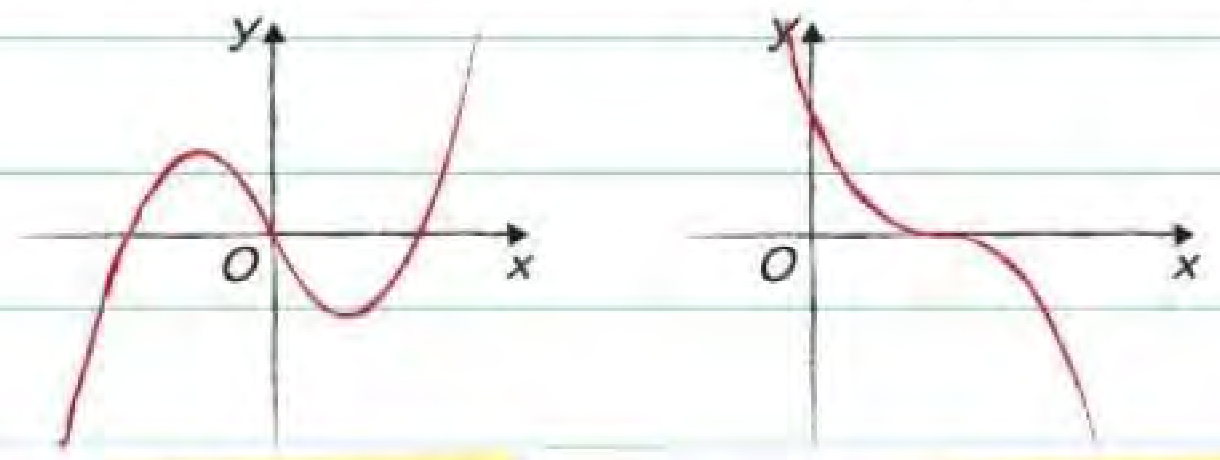
Factorise then sketch

You can sketch the graphs of cubic and quartic functions by **factorising** them to find their roots.



Shapes and roots

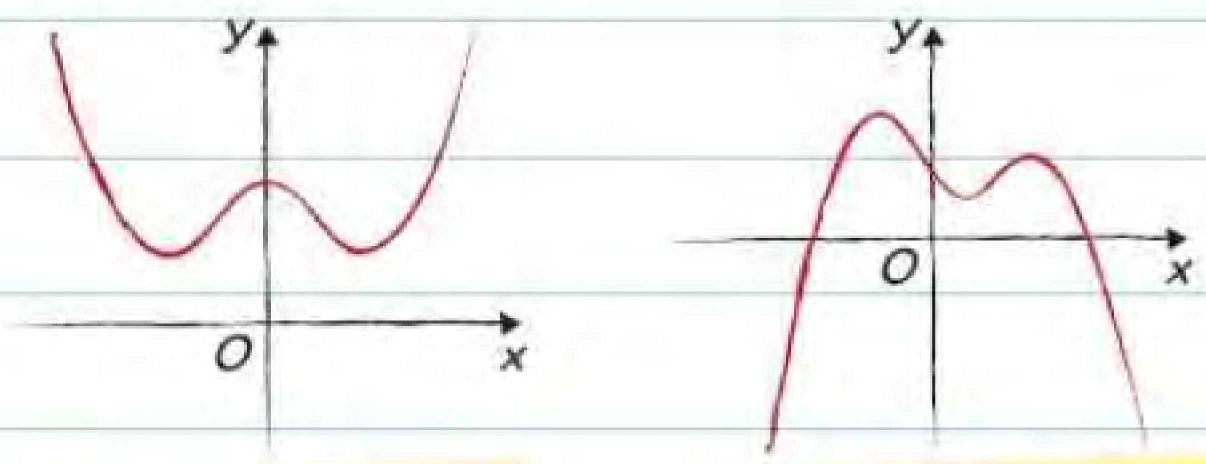
Cubic functions can have 1, 2 or 3 real roots:



Positive x^3 term and 3 real roots

Negative x^3 term and 1 repeated real root

Quartic functions can have 0, 1, 2, 3 or 4 real roots:



Positive x^4 term and 0 real roots

Negative x^4 term and 2 real roots

Considering infinity

In the example on the right, as x gets large, the x^3 term gets large **more quickly** than the x^2 term. So for large positive x, y gets very large. You can write 'as $x \to \infty$, $y \to \infty$'.

Similarly, as $x \to -\infty$, $y \to -\infty$.

This tells you how the curve will behave at either end of the x-axis.

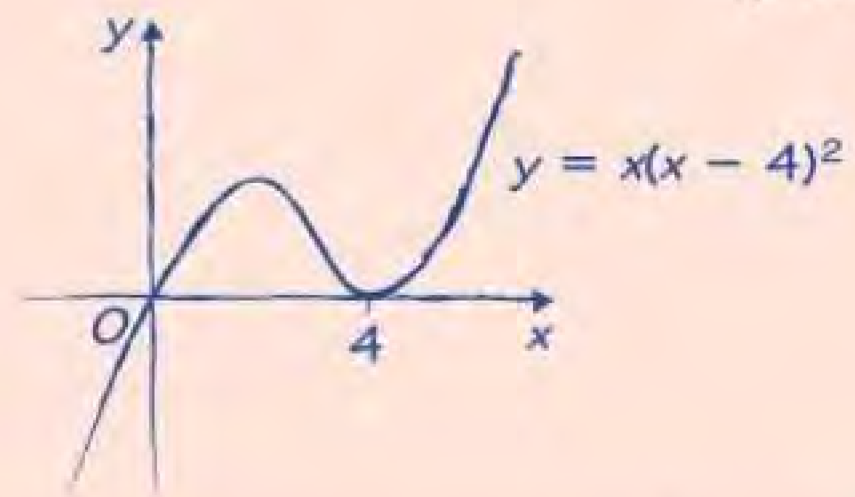
The factorised equation has a factor of x so the curve will pass through the origin. It also has a **repeated** factor of $(x - 4)$ so the curve will just touch the x-axis at the point $x = 4$.

Worked example

(a) Factorise completely $x^3 - 8x^2 + 16x$ **(3 marks)**

$x(x^2 - 8x + 16) = x(x - 4)^2$

(b) Hence sketch the curve with equation $y = x^3 - 8x^2 + 16x$, showing the points where the curve meets the coordinate axes. **(3 marks)**



Now try this

Answer

1 (a) Factorise completely $x^3 - 9x$ **(3 marks)**

(b) Hence sketch the curve $y = x^3 - 9x$

Answer **(3 marks)**

2 Sketch the graph of $y = (2x - 1)(x - 3)^2$, showing clearly the coordinates of the points where the curve meets the coordinate axes. **(4 marks)**

Answer

3 Sketch the graph of $y = x(5 - x)(2x^2 + 9x + 4)$. Show clearly the coordinates of any points where the curve meets or crosses the coordinate axes. Answer **(4 marks)**

You need to show the coordinates of the point where the graph meets the y-axis as well.

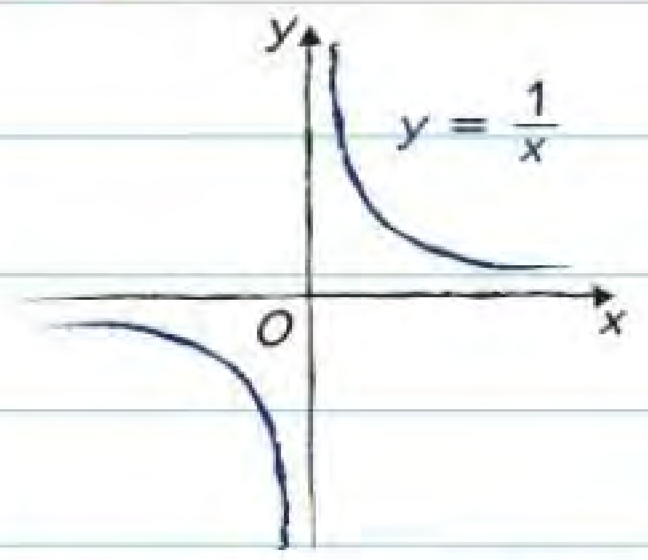
Reciprocal graphs

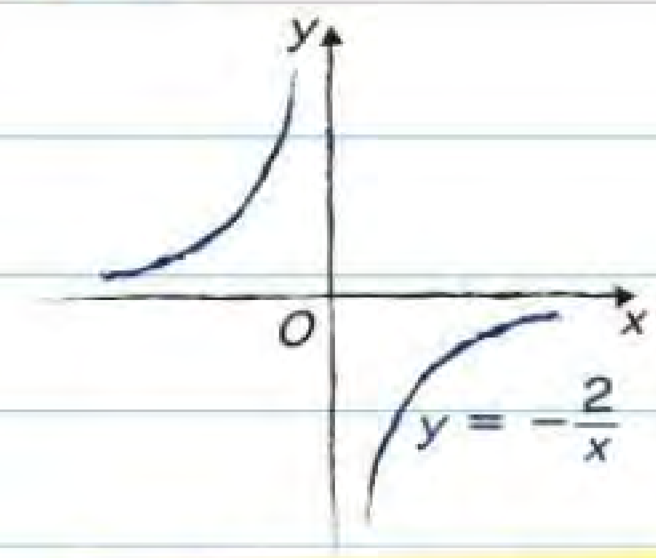
You need to know how to sketch the graphs of $y = \frac{k}{x}$ and $y = \frac{k}{x^2}$, and transformations of these graphs.

Shapes and asymptotes

The shapes of the reciprocal graphs are different for **positive** and **negative** values of k:

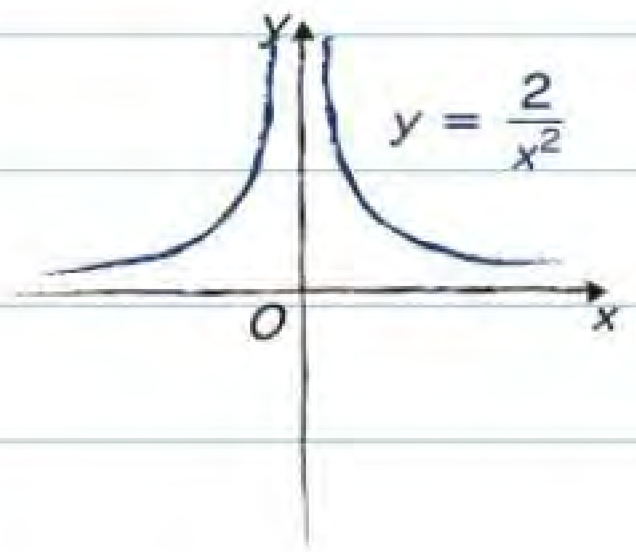
1 $k > 0$ **2** $k < 0$

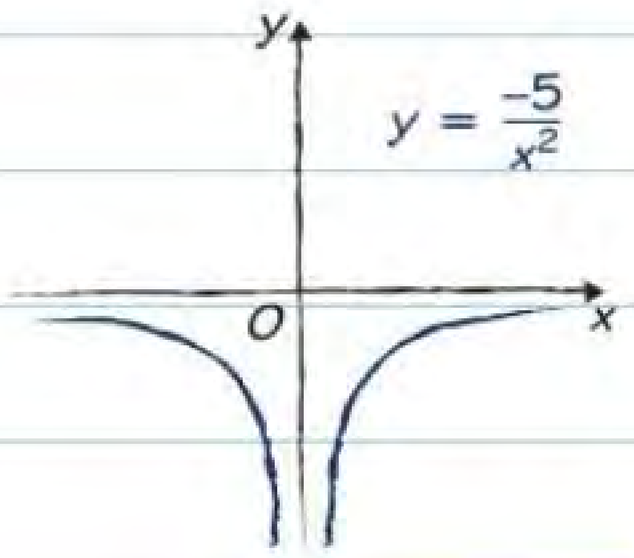




$y = \frac{k}{x}$ with positive k

$y = \frac{k}{x}$ with negative k





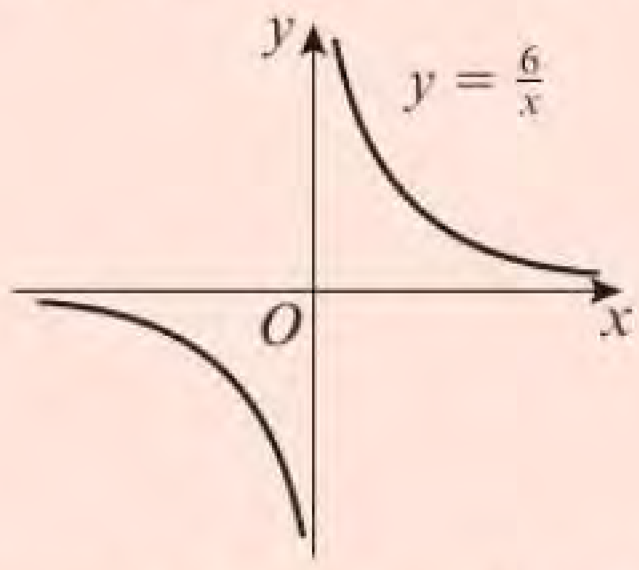
$y = \frac{k}{x^2}$ with positive k

$y = \frac{k}{x^2}$ with negative k

The graphs of $y = \frac{k}{x}$ and $y = \frac{k}{x^2}$ have **asymptotes** at the x-axis and the y-axis. Remember to translate any asymptotes when you translate the graph.

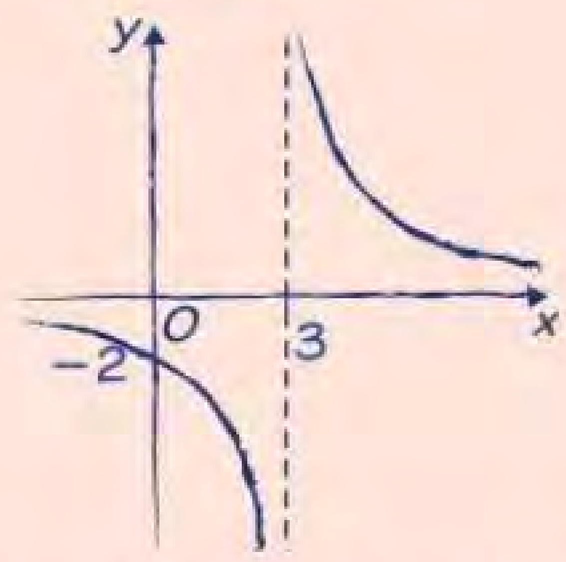
Worked example

The figure shows a sketch of the curve $y = \frac{6}{x}$



(a) On a separate diagram, sketch the curve with equation $y = \frac{6}{x-3}$, showing any points at which the curve crosses the coordinate axes. **(3 marks)**

When $x = 0$, $y = \frac{6}{-3} = -2$



(b) Write down the equation of the asymptotes of the curve in part (a). **(2 marks)**

$y = 0$ and $x = 3$

The transformation from $y = \frac{6}{x}$ to $y = \frac{6}{x-3}$ is the translation $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

Now try this

1 Sketch the graph of $y = -\frac{4}{x}$ **(2 marks)**

Answer

The transformation is $y = f(x) \rightarrow y = f(x + 1)$
Draw the new asymptote on your sketch before you draw your curve.

2 (a) Sketch the graph of $y = \frac{3}{x}$ **(2 marks)**

Answer

(b) On a separate diagram, sketch the graph of $y = \frac{3}{x+1}$, showing any points at which the curve crosses the coordinate axes. **(3 marks)**

Answer

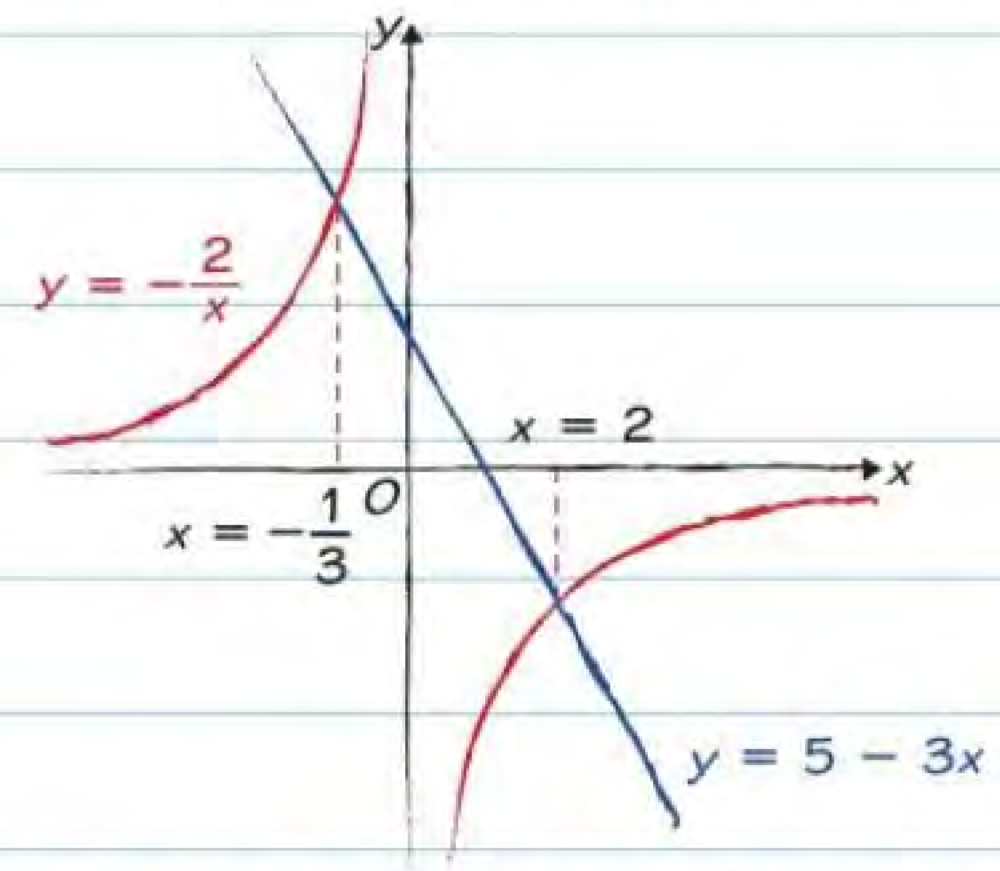
(c) Write down the equations of the asymptotes of the curve in part (b). **(2 marks)**

Answer

Points of intersection

The coordinates of the points where two graphs **intersect** are the x- and y-values which satisfy **both** equations at the same time. You can use algebra to find the points where two curves intersect.

The diagram shows the graphs of $y = -\frac{2}{x}$ and $y = 5 - 3x$.



The x-coordinates at the points of intersection are the solutions to the equation

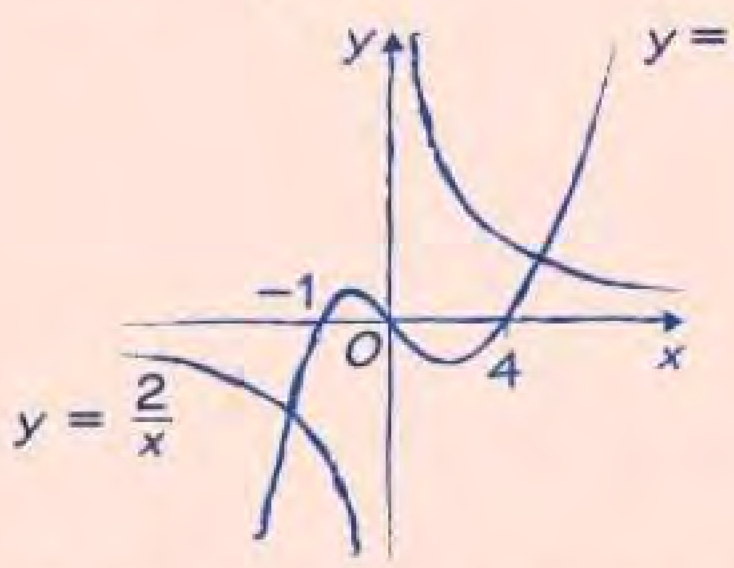
$$5 - 3x = -\frac{2}{x}$$
$$x(5 - 3x) = -2$$
$$5x - 3x^2 = -2$$
$$3x^2 - 5x - 2 = 0$$
$$(3x + 1)(x - 2) = 0$$
$$x = -\frac{1}{3} \quad \text{or} \quad x = 2$$

Worked example

(a) On the same axes, sketch the graph with the equation

(i) $y = x(x + 1)(x - 4)$

(ii) $y = \frac{2}{x}$ **(5 marks)**



(b) Write down the number of real solutions to the equation $x(x + 1)(x - 4) = \frac{2}{x}$ **(1 mark)**

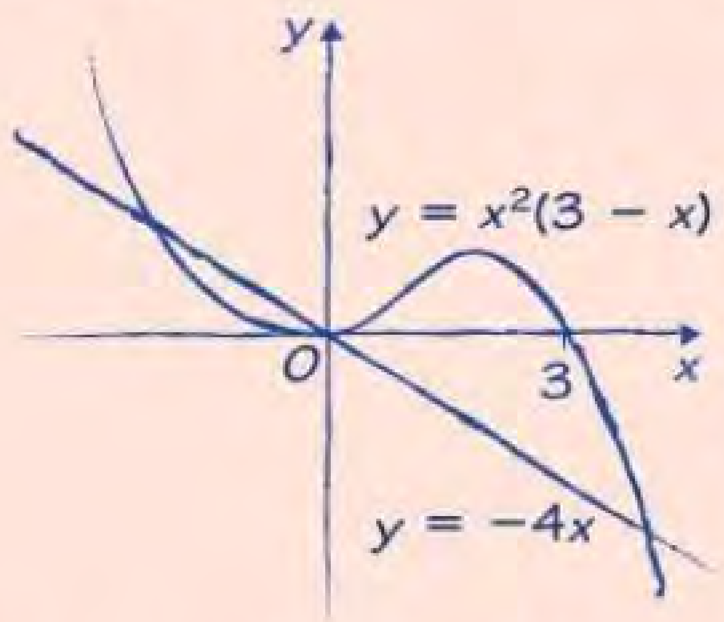
2

Worked example

(a) On the same axes, sketch the graphs with the equations

(i) $y = x^2(3 - x)$

(ii) $y = -4x$ **(5 marks)**



(b) Find the coordinates of the points of intersection. **(6 marks)**

$$3x^2 - x^3 = -4x$$
$$x^3 - 3x^2 - 4x = 0$$
$$x(x^2 - 3x - 4) = 0$$
$$x(x - 4)(x + 1) = 0$$
$$x = 0 \quad \text{or} \quad x = 4 \quad \text{or} \quad x = -1$$
$$y = 0 \qquad y = -16 \qquad y = 4$$
$$(0, 0),\ (4, -16),\ (-1, 4)$$

The points of intersection will be solutions to the equation $x^2(3 - x) = -4x$.

Now try this

(a) On the same axes, sketch the graph with the equation

(i) $y = x^2(x - 3)$ Answer

(ii) $y = x(8 - x)$ **(6 marks)**

Indicate all the points where the curves meet the x-axis.

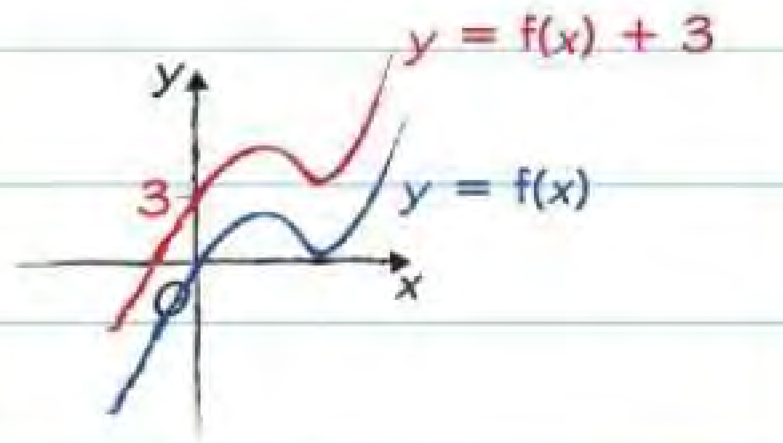
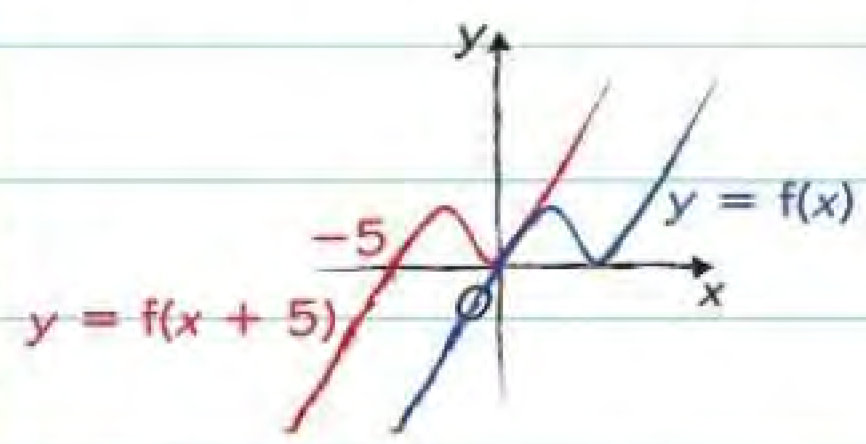
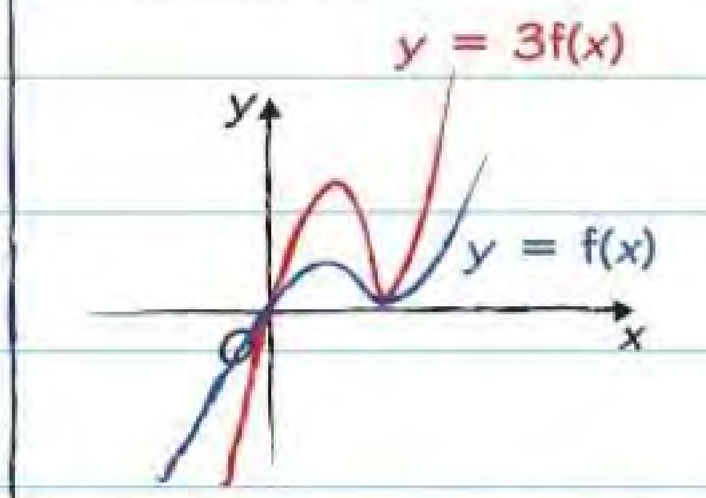
(b) Use algebra to find the coordinates of the points of intersection. **(7 marks)**

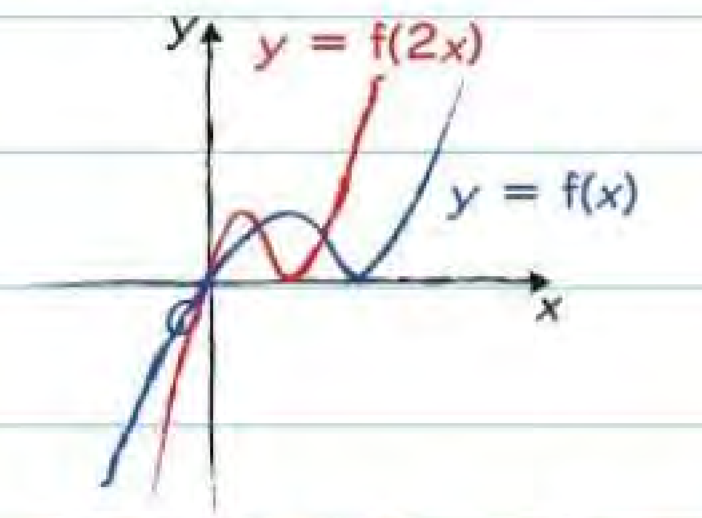
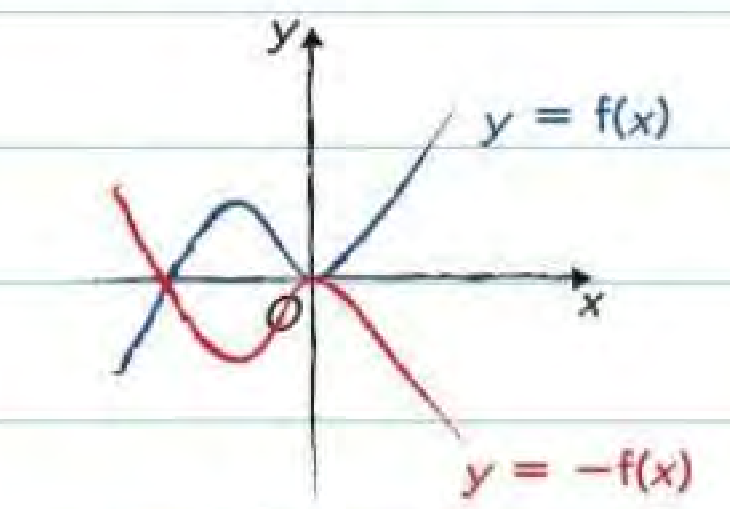
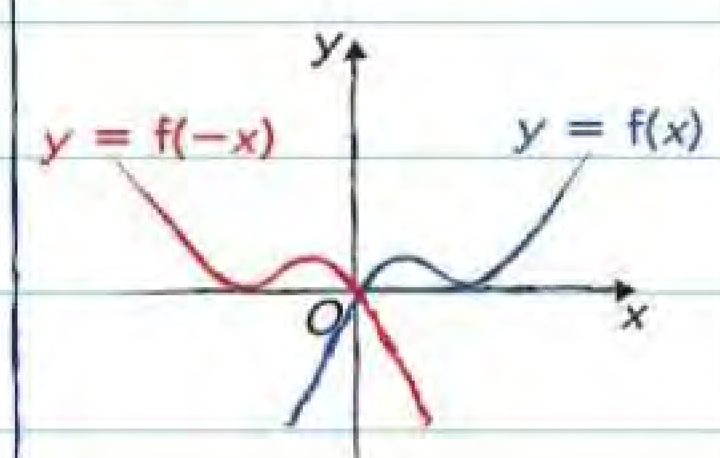
Answer

There are three points of intersection: one at (0, 0), one with a negative value of x and one with a positive value of x.

Transformations 1

You can change the equation of a graph to translate it, stretch it or reflect it. These tables show you how you can use functions to transform the graph of $y = \mathrm{f}(x)$.

Function	$y = \mathrm{f}(x) + a$	$y = \mathrm{f}(x + a)$	$y = a\mathrm{f}(x)$
Transformation of graph	Translation $\begin{pmatrix} 0 \\ a \end{pmatrix}$	Translation $\begin{pmatrix} -a \\ 0 \end{pmatrix}$	Stretch in the vertical direction, scale factor a
Useful to know	$\mathrm{f}(x) + a \rightarrow$ move UP a units $\mathrm{f}(x) - a \rightarrow$ move DOWN a units	$\mathrm{f}(x + a) \rightarrow$ move LEFT a units $\mathrm{f}(x - a) \rightarrow$ move RIGHT a units	x-values stay the same
Example			

Function	$y = \mathrm{f}(ax)$	$y = -\mathrm{f}(x)$	$y = \mathrm{f}(-x)$
Transformation of graph	Stretch in the horizontal direction, scale factor $\frac{1}{a}$	Reflection in the x-axis	Reflection in the y-axis
Useful to know	y-values stay the same	'−' outside the bracket	'−' inside the bracket
Example			

Worked example

The diagram shows a sketch of a curve with equation $y = \mathrm{f}(x)$.

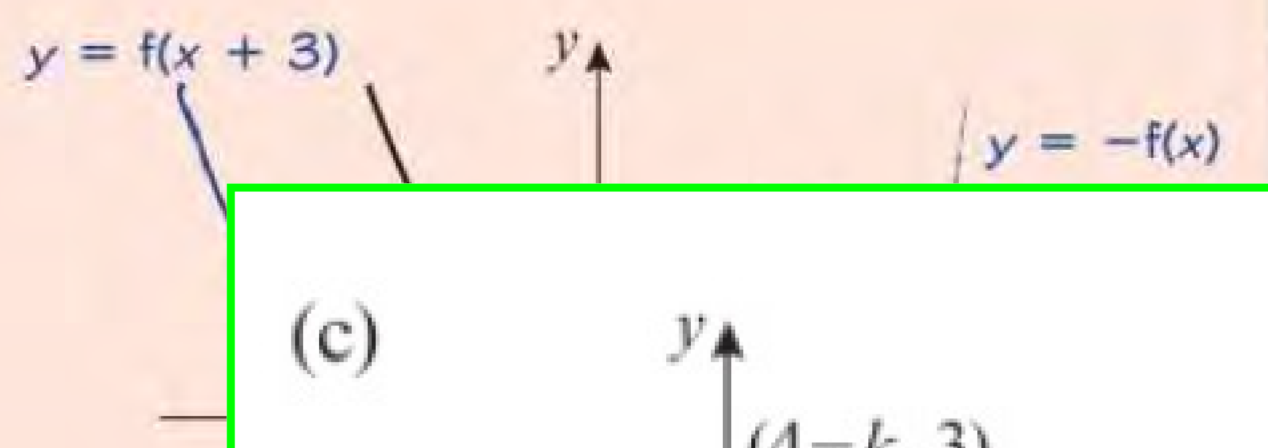
On the same diagram sketch the curve with equation

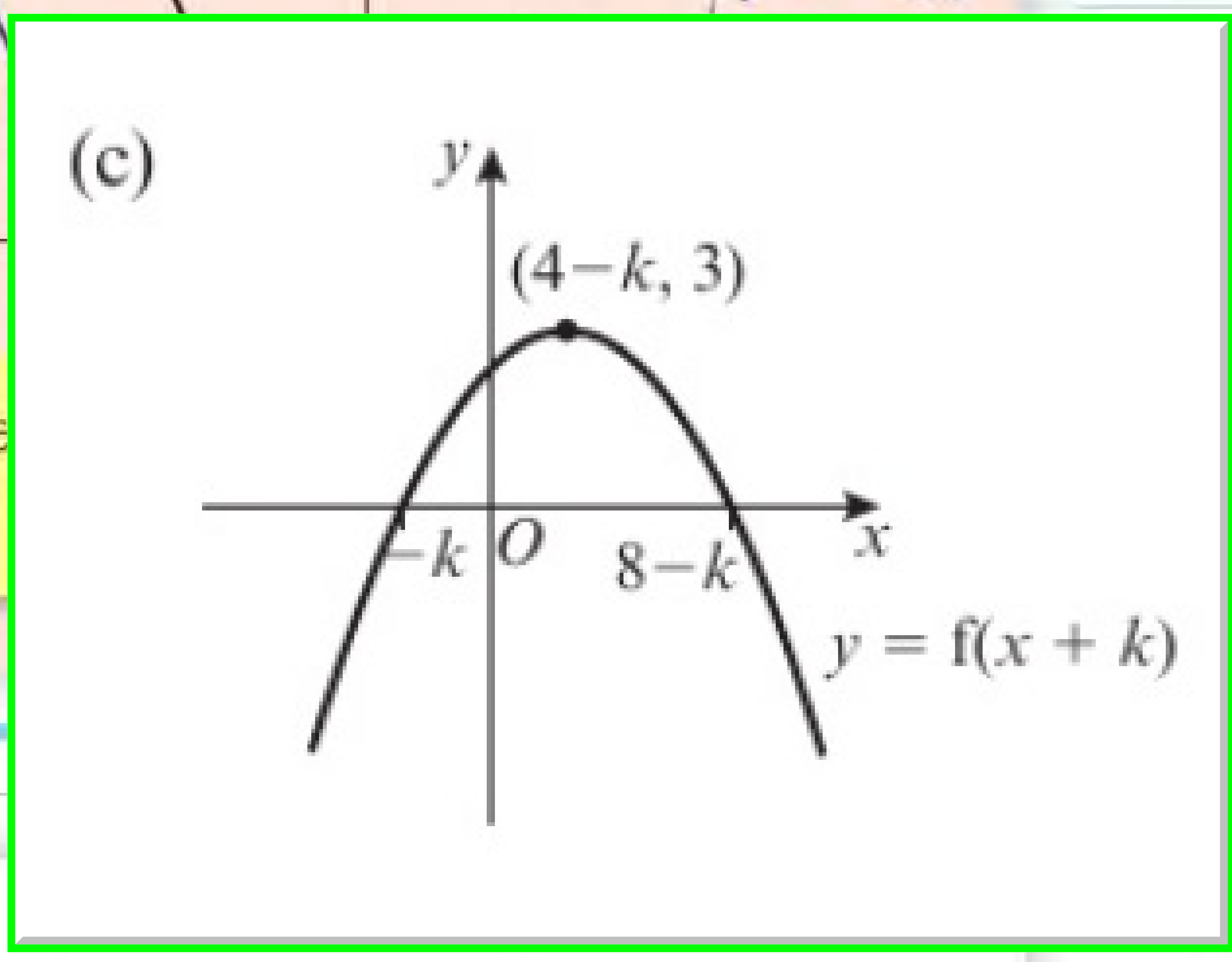
(a) $y = \mathrm{f}(x + 3)$ **(3 marks)**

(b) $y = -\mathrm{f}(x)$. **(3 marks)**

Show clearly the coordinates of any maximum or minimum points, and any points of intersection with the axes.

Everything in blue is part of the answer.





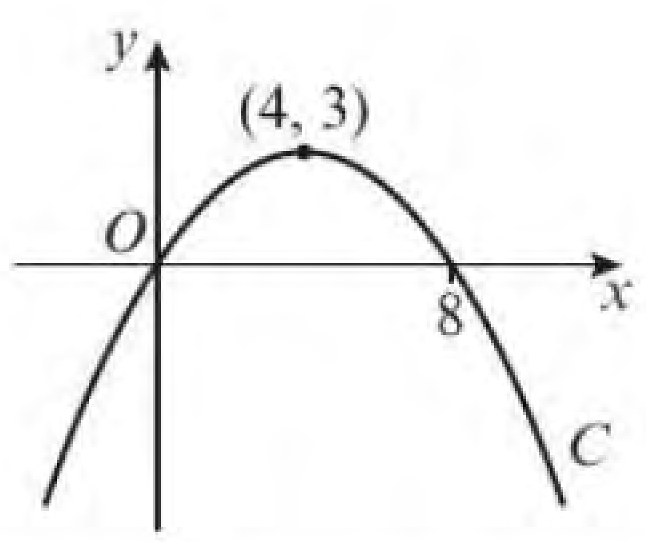
Now try this

The diagram shows a sketch of a curve C with equation $y = \mathrm{f}(x)$.
On separate diagrams sketch the curve with equation

(a) $y = 2\mathrm{f}(x)$ **(3 marks)** Answer (b) $y = \mathrm{f}(-x)$ Answer **(3 marks)**

(c) $y = \mathrm{f}(x + k)$, where k is a constant and $0 < k < 4$ Answer **(4 marks)**

On each diagram show the coordinates of any maximum or minimum points, and any points of intersection with the x-axis.

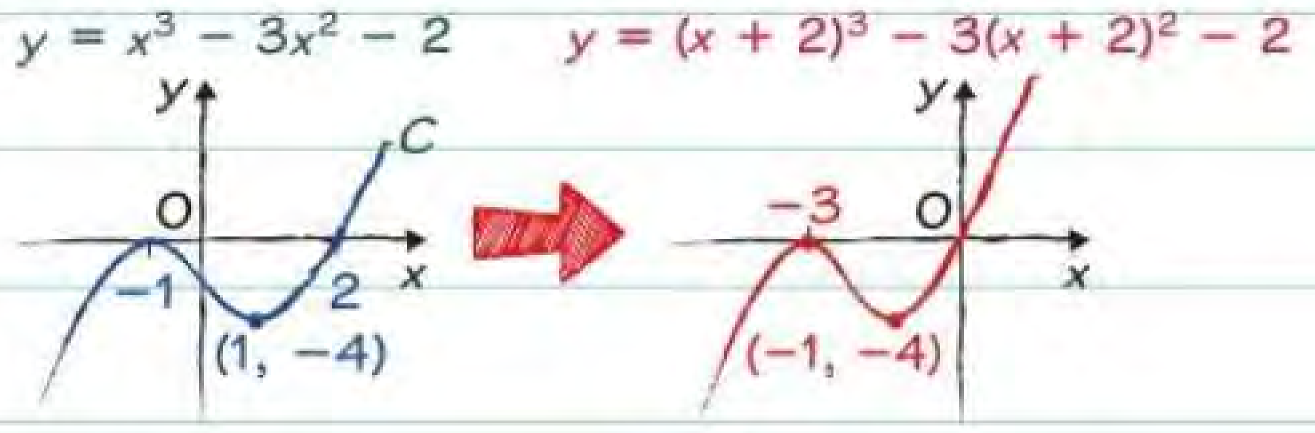


Transformations 2

You need to be able to spot transformed functions from their equations, and sketch transformations involving **asymptotes**.

Functions and equations

Curve C below has equation $y = x^3 - 3x^2 - 2$. You can sketch the curves of other equations by transforming curve C.

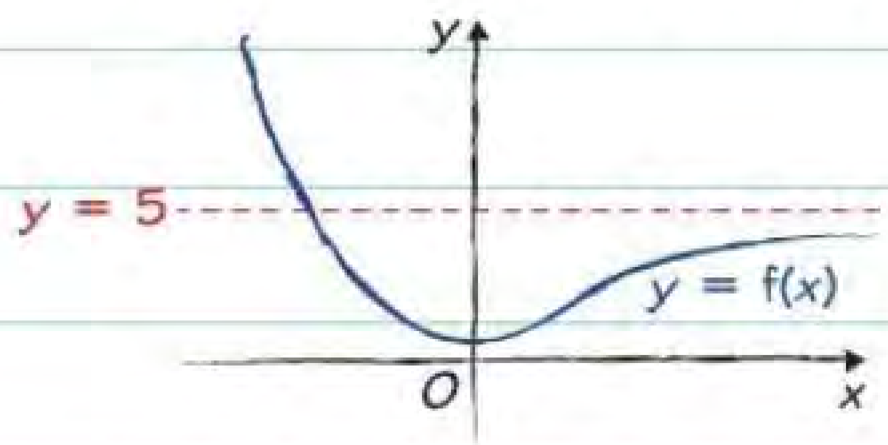


$y = f(x)$
$y = x^3 - 3x^2 - 2$

$y = f(x + 2)$
$y = (x + 2)^3 - 3(x + 2)^2 - 2$

Asymptotes

An asymptote is a line which a curve approaches, but never reaches. You draw asymptotes on graphs with DOTTED LINES.



This curve has an asymptote at $y = 5$. When you transform a graph, its asymptotes are transformed as well.

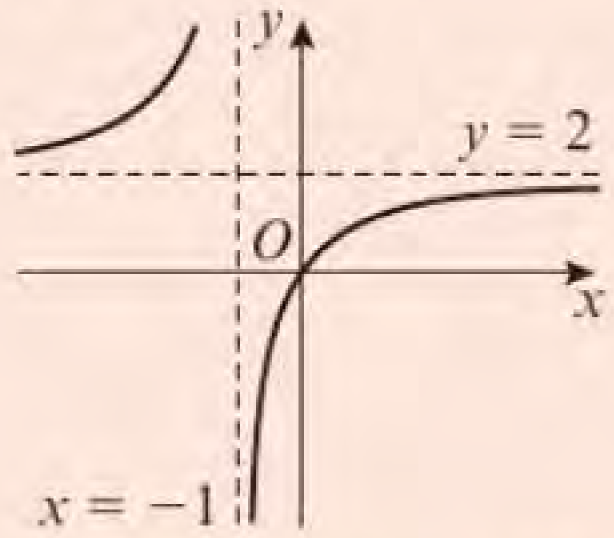
Transformation	New asymptote
$y = f(x) - 1$	$y = 4$
$y = 2f(x)$	$y = 10$
$y = f(x + 4)$	$y = 5$

The graph is translated 4 units to the left, so the horizontal asymptote does not change.

Worked example

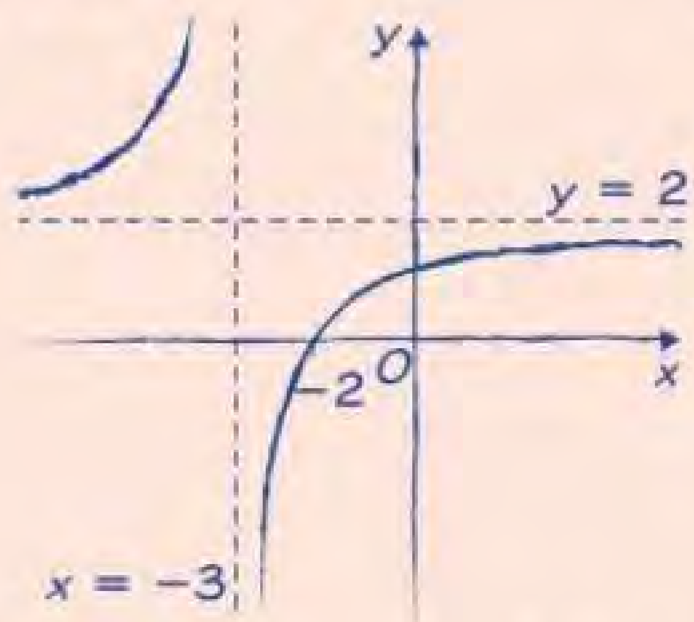
The diagram shows a sketch of the curve with equation $y = f(x)$ where

$$f(x) = \frac{2x}{x + 1},\ x \neq -1$$



The curve has asymptotes with equations $y = 2$ and $x = -1$

(a) Sketch the curve with equation $y = f(x + 2)$ and state the equations of its asymptotes. **(3 marks)**



(b) Find the coordinates of the points where the curve in part (a) crosses the coordinate axes. **(3 marks)**

$$f(x + 2) = \frac{2(x + 2)}{(x + 2) + 1} = \frac{2x + 4}{x + 3}$$

When $x = 0$, $y = \frac{4}{3}$

$(-2, 0)$ and $(0, \frac{4}{3})$

Now try this

The diagram shows a curve C with equation $y = f(x)$, where

$$f(x) = \frac{(x + 2)^2}{x + 1},\ x \neq -1$$

(a) Sketch the curve with equation $y = f(x + 1)$ and state the new equation of the asymptote $x = -1$ Answer **(3 marks)**

(b) Write down the coordinates of the points where the curve meets the coordinate axes. Answer **(3 marks)**