Summary of key points

6 If $f'(x)$ is the derivative of $f(x)$ for all values of x in the interval $[a, b]$, then the definite integral is defined as $\int_a^b f'(x)\,dx = [f(x)]_a^b = f(b) - f(a)$

7 The area between a positive curve, the x-axis and the lines $x = a$ and $x = b$ is given by

$$\text{Area} = \int_a^b y\,dx$$

where $y = f(x)$ is the equation of the curve.

8 When the area bounded by a curve and the x-axis is below the x-axis, $\int y\,dx$ gives a negative answer.

9 You can use definite integration together with areas of trapeziums and triangles to find more complicated areas on graphs.

- Integrate the function
- Substitute the values (x, y) of a point on the curve, or the value of the function at a given point $f(x) = k$ into the integrated function
- Solve the equation to find c

Definite integration

In your exam, you might have to find an integral with **limits**. This is called definite integration. You should make sure you are confident with **indefinite integration** before revising this – check page 42 for a reminder.

Evaluating a definite integral

A definite integral has a **numerical answer**.

upper limit of the integral

lower limit of the integral

Integrate $(6x + 1)$ in the normal way and write the integral in **square brackets**. You can **ignore** the constant of integration.

Write the limits next to the square brackets.

$$\int_2^5 (6x + 1)dx = [3x^2 + x]_2^5$$
$$= (3 \times 5^2 + 5) - (3 \times 2^2 + 2)$$
$$= 80 - 14$$
$$= 66$$

evaluate the integral at the **upper limit**...

... and **subtract** the value of the integral at the **lower limit**.

Worked example

Use calculus to find the value of

$\int_1^9 (2x + 6\sqrt{x})\,dx$ **(5 marks)**

$$\int_1^9 \left(2x + 6x^{\frac{1}{2}}\right)dx = \left[x^2 + 4x^{\frac{3}{2}}\right]_1^9$$
$$= \left(9^2 + 4 \times 9^{\frac{3}{2}}\right) - \left(1^2 + 4 \times 1^{\frac{3}{2}}\right)$$
$$= 189 - 5$$
$$= 184$$

Some calculators have a key like the one shown here which can work out numerical integration.

You **cannot** just use this key and write down your answer. You need to **use calculus** and **show your working**, or you will get a maximum of 1 mark. You could use this key to check your answer, but it's safer to stay away from it entirely!.

If you are using calculus to do definite integration you will usually need to give an exact answer. If your answer isn't a whole number or a fraction, write it in simplified **surd form**.

$2\sqrt{12} = 2\sqrt{4 \times 3} = 2\sqrt{4}\sqrt{3} = 4\sqrt{3}$

Have a look at page 3 for a reminder about surds.

Worked example

Find the exact value of $\int_1^{12} \left(\frac{1}{\sqrt{x}}\right)dx$ **(4 marks)**

$$\int_1^{12} \left(x^{-\frac{1}{2}}\right)dx = \left[2x^{\frac{1}{2}}\right]_1^{12}$$
$$= (2\sqrt{12}) - (2\sqrt{1})$$
$$= 4\sqrt{3} - 2$$

Now try this

Start by writing $\frac{6}{x^2}$ as $6x^{-2}$.

1 Use calculus to find the exact value of

$\int_1^3 \left(3x^2 - 7 + \frac{6}{x^2}\right)dx$ Answer **(5 marks)**

2 Given that $f(x) = \frac{6}{x^3} - \frac{2}{\sqrt{x}}$, find $\int_1^2 f(x)\,dx$, giving your answer in the form $a - b\sqrt{2}$, where a and b are constants. **(5 marks)**

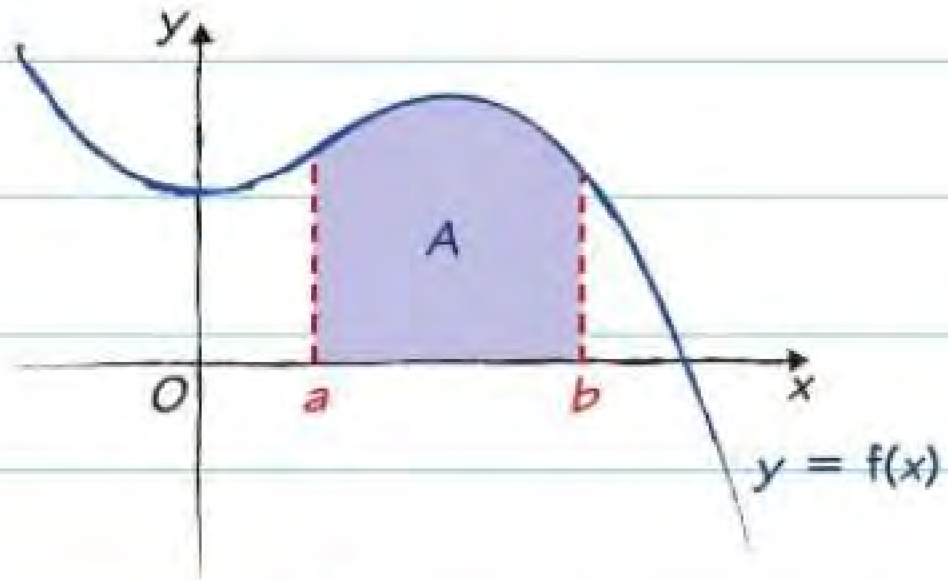
Answer

Be careful when you are subtracting the value of the integral at $x = 1$. Use brackets to make sure you don't make a mistake with negative numbers.

Area under a curve

You can use **definite integration** to find the area between a curve and the x-axis. The area between the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is given by

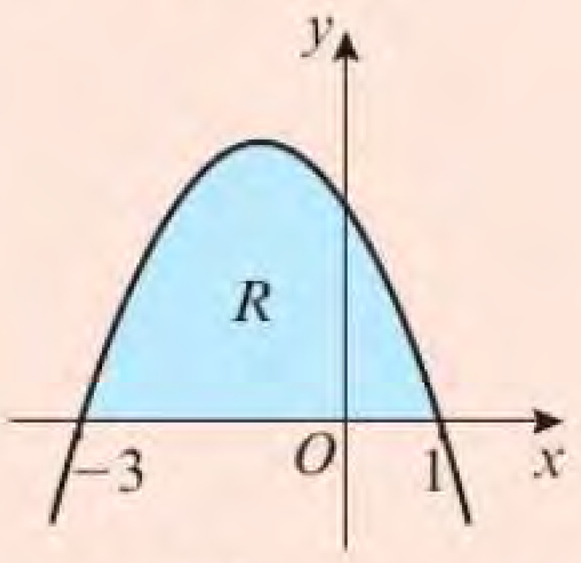
$$A = \int_a^b f(x)\,dx$$



Look at the previous page for a reminder about definite integration.

Worked example

The diagram shows part of the curve with equation $y = (x + 3)(1 - x)$



Use calculus to find the exact area of the shaded region, R. **(5 marks)**

$y = 3 - 2x - x^2$

$\int_{-3}^{1}(3 - 2x - x^2)\,dx = \left[3x - x^2 - \frac{x^3}{3}\right]_{-3}^{1}$

$= (3 - 1 - \frac{1}{3}) - (-9 - 9 + 9)$

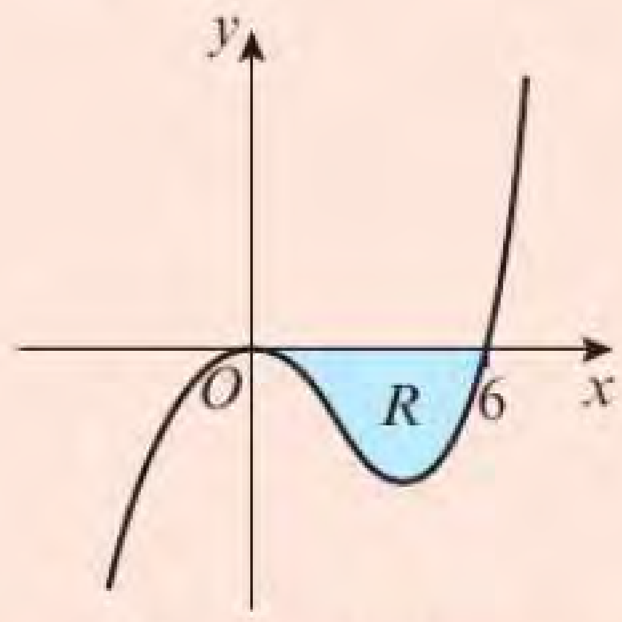
$= \frac{5}{3} - (-9) = 10\frac{2}{3}$

Area of $R = 10\frac{2}{3}$

The curve crosses the x-axis at -3 and 1. So the **limits** for your **definite integral** will be -3 and 1. Always put the **right-hand** boundary as the **upper limit** and the left-hand boundary as the **lower limit**. You can give your exact answer as a fraction, mixed number or decimal.

Worked example

The diagram shows part of the curve with equation $y = x^2(x - 6)$



Find the area of the shaded region, R. **(6 marks)**

$y = x^3 - 6x^2$

$\int_{0}^{6}(x^3 - 6x^2)\,dx = \left[\frac{x^4}{4} - 2x^3\right]_{0}^{6}$

$= (324 - 432) - (0 - 0)$

$= -108$

Area of $R = 108$

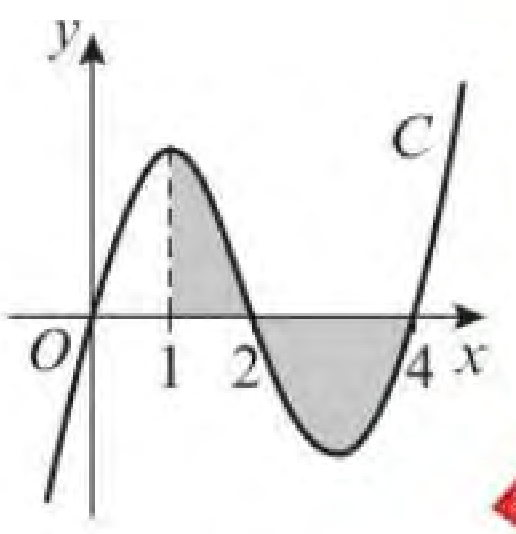
Negative areas

When you use a **definite integral** to find an area **below** the x-axis, the answer will be **negative**. If you are asked to find an area, make sure you give your final answer as a **positive** number.

Now try this

The diagram shows part of the curve C with equation $y = x(x - 2)(x - 4)$

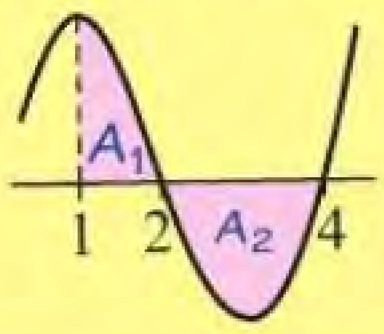
Use calculus to find the total area of the shaded region, between $x = 1$ and $x = 4$ and bounded by C, the x-axis, and the line $x = 1$



(9 marks)

Answer

Be careful with this question – you can't just find $\int_1^4 y\,dx$ because an area **below** the x-axis will produce a **negative** integral. You need to work out **two separate** definite integrals to find these two areas, then add the areas together:



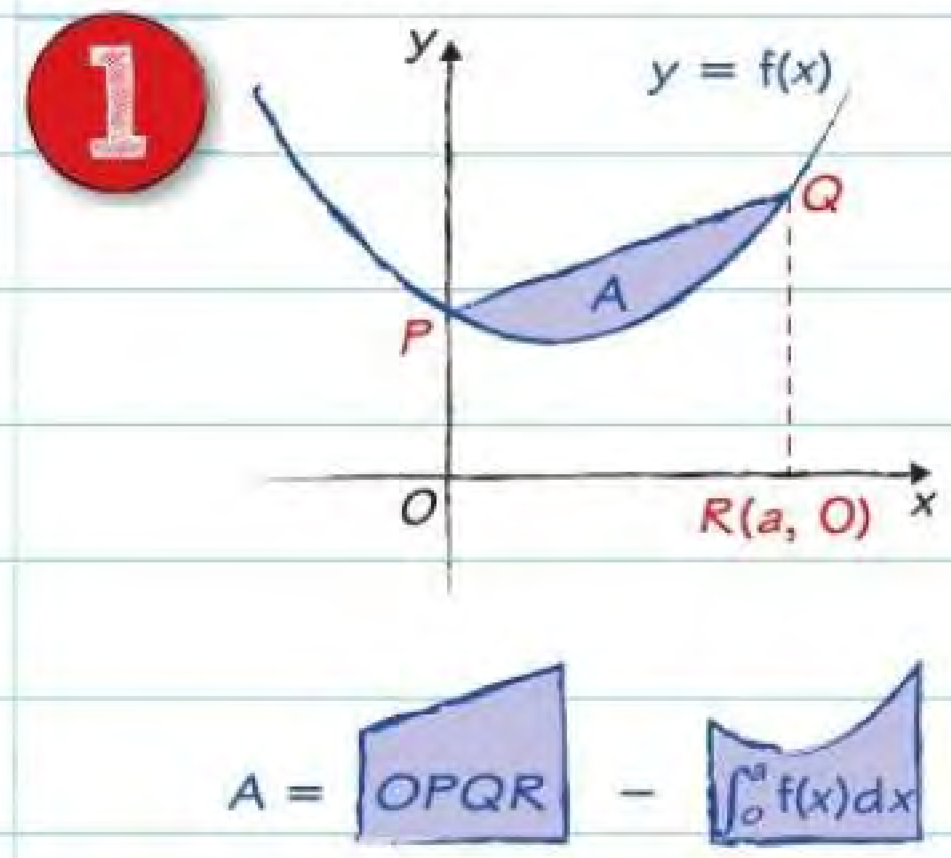
$A_1 = \int_1^2 y\,dx$

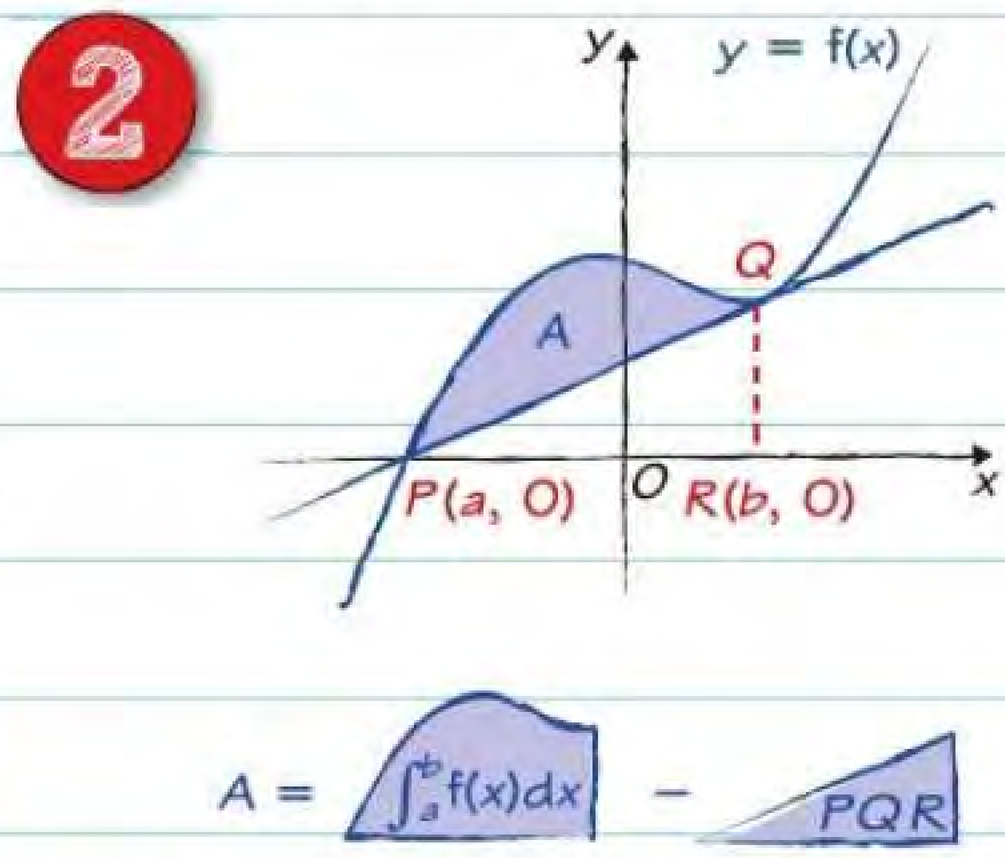
$A_2 = -\int_2^4 y\,dx$

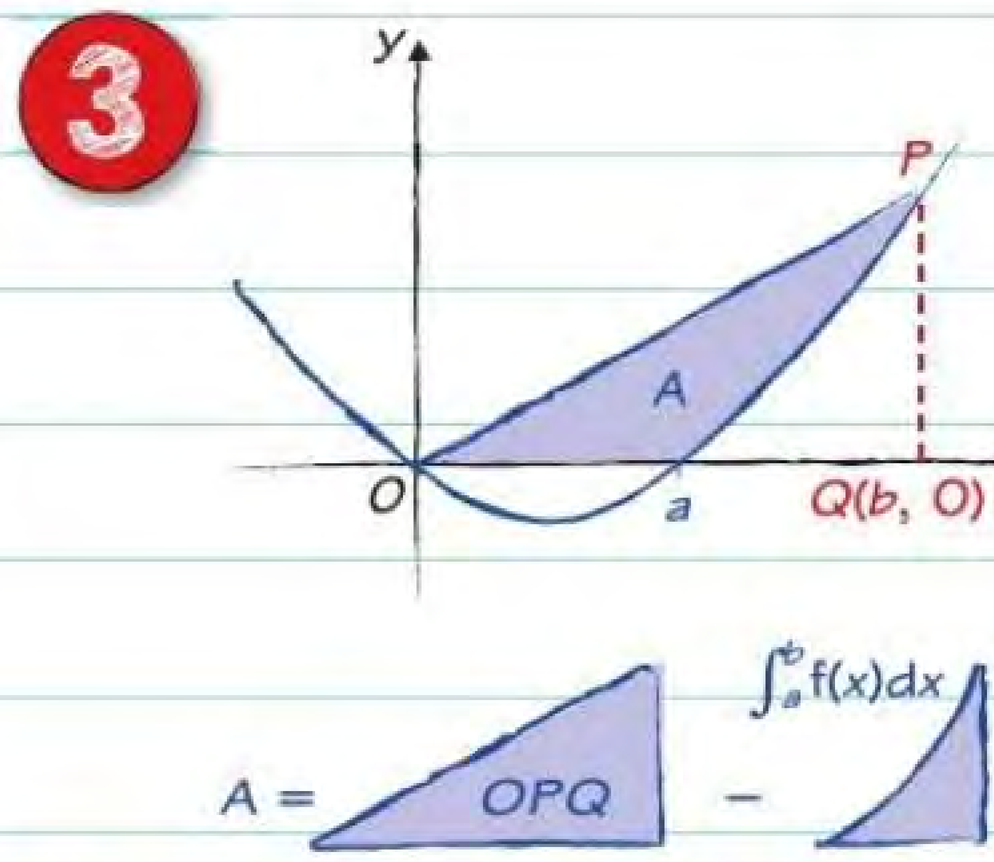
Total area $= A_1 + A_2$

More areas

You can use areas of **triangles** and **trapeziums**, together with **definite integration**, to find areas enclosed by curves and straight lines. Here are three examples.

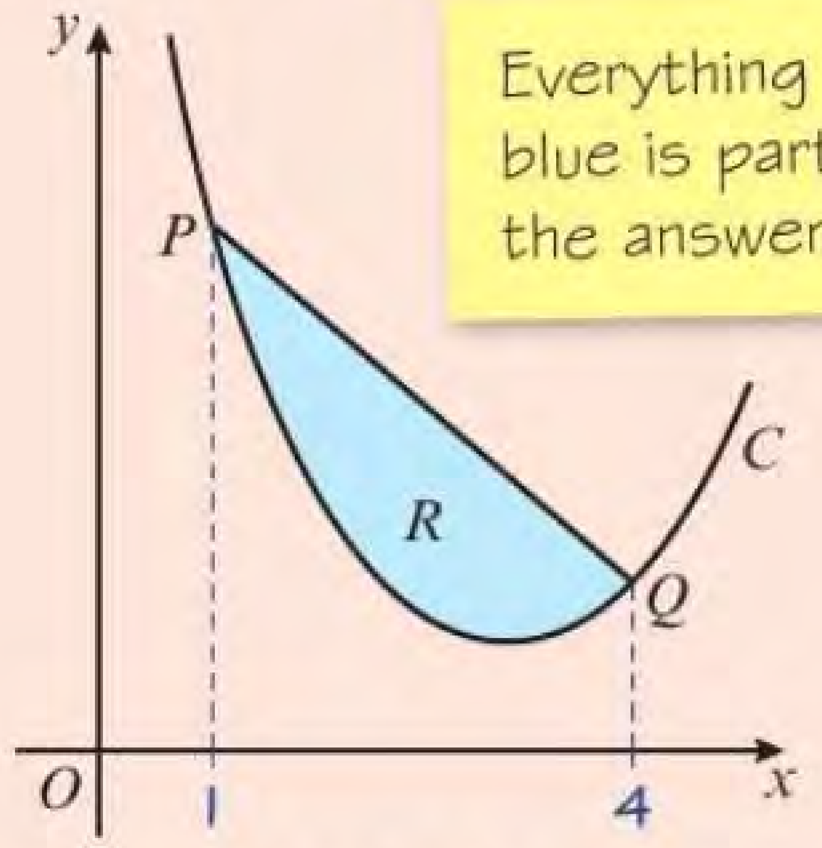






Worked example

The diagram shows part of the curve C with equation $y = \frac{8}{x^2} + x - 2, x > 0$



Everything in blue is part of the answer.

The points P and Q lie on the curve and have x-coordinates 1 and 4 respectively.
The region R is bounded by the curve and the line segment PQ. Find the exact area of R.

(8 marks)

When $x = 1$, $y = 8 + 1 - 2 = 7$

So coordinates of P are (1, 7).

When $x = 4$, $y = \frac{1}{2} + 4 - 2 = 2\frac{1}{2}$

So coordinates of Q are (4, 2.5).

Area of trapezium $= \frac{1}{2}(7 + 2.5)(4 - 1)$

$= 14.25$

$\int_1^4 (8x^{-2} + x - 2)\,dx = \left[-8x^{-1} + \frac{1}{2}x^2 - 2x\right]_1^4$

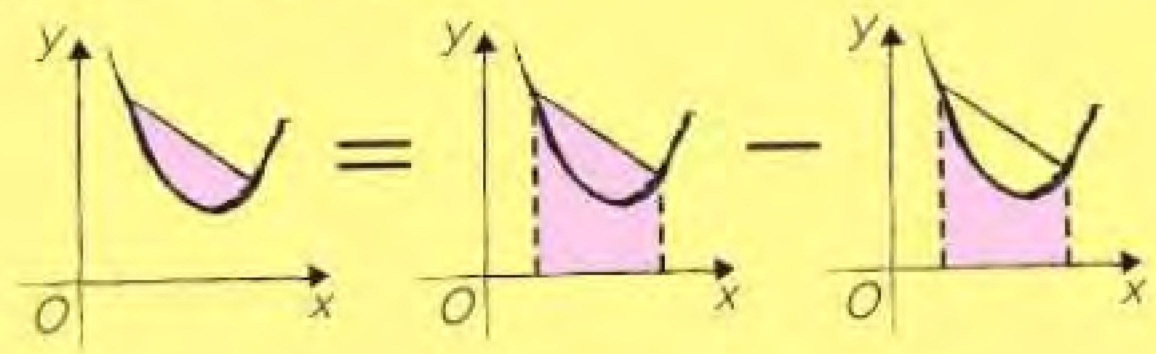
$= (-2 + 8 - 8) - (-8 + \frac{1}{2} - 2)$

$= -2 - (-9\frac{1}{2})$

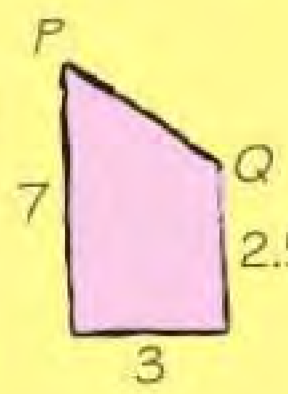
$= 7.5$

So area of $R = 14.25 - 7.5 = 6.75$.

If you have to find the area between a curve and a line like this, **plan your answer** before you start. Work out how you can use triangles, trapeziums and rectangles to work out the area.



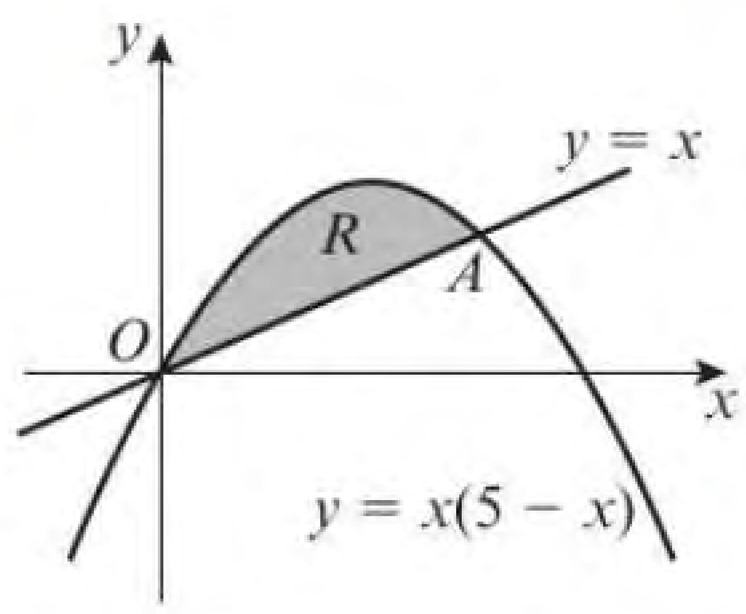
Before you can work out the area of the trapezium, you need to know the y-coordinates of P and Q. It's sometimes easier to work in decimals rather than fractions.



Use the formula $A = \frac{1}{2}(a + b)h$ to work out the area of the trapezium. Then work out $\int_1^4 y\,dx$ and subtract it from the area of the trapezium, to find the shaded area.

Now try this

The straight line with equation $y = x$ cuts the curve with equation $y = x(5 - x)$ at the points O and A.



(a) Find the coordinates of A. **(2 marks)** Answer

(b) Use calculus to find the exact area of the shaded region R. **(7 marks)** Answer