Summary of key points

1 You can use **Euler's relation**, $e^{i\theta} = \cos\theta + i\sin\theta$, to write a complex number z in exponential form:

$$z = re^{i\theta}$$

where $r = |z|$ and $\theta = \arg z$.

Note Substituting $\theta = \pi$ into Euler's relation yields **Euler's identity**:

$$\mathbf{e^{i\pi} + 1 = 0}$$

This equation links the five fundamental constants 0, 1, π, e and i, and is considered an example of mathematical beauty.

2 For any two complex numbers $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$,

- $z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}$
- $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$

Problem-solving

$\cos\theta = \cos(\theta + 2\pi)$ and $\sin\theta = \sin(\theta + 2\pi)$.

Subtract multiples of 2π from $\dfrac{23\pi}{5}$ until you find a value in the range $-\pi < \theta \leqslant \pi$.

3 **De Moivre's theorem:**

For any integer n, $(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta)$

4
- $z + \dfrac{1}{z} = 2\cos\theta$
- $z^n + \dfrac{1}{z^n} = 2\cos n\theta$
- $z - \dfrac{1}{z} = 2i\sin\theta$
- $z^n - \dfrac{1}{z^n} = 2i\sin n\theta$

Notation In exponential form, these results are equivalent to:

$$\cos n\theta = \frac{1}{2}(e^{in\theta} + e^{-in\theta}) \qquad \sin n\theta = \frac{1}{2i}(e^{in\theta} - e^{-in\theta}).$$

5 For $w, z \in \mathbb{C}$,

- $\displaystyle\sum_{r=0}^{n-1} wz^r = w + wz + wz^2 + \ldots + wz^{n-1} = \frac{w(z^n - 1)}{z - 1}$
- $\displaystyle\sum_{r=0}^{\infty} wz^r = w + wz + wz^2 + \ldots = \frac{w}{1-z}, \ |z| < 1$

6 If z and w are non-zero complex numbers and n is a positive integer, then the equation $z^n = w$ has n distinct solutions.

7 For any complex number $z = r(\cos\theta + i\sin\theta)$, you can write

$$z = r(\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi))$$

where k is any integer.

8 In general, the solutions to $z^n = 1$ are $z = \cos\left(\dfrac{2\pi k}{n}\right) + i\sin\left(\dfrac{2\pi k}{n}\right) = e^{\frac{2\pi ik}{n}}$ for $k = 1, 2, \ldots, n$ and are known as the nth roots of unity.

If n is a positive integer, then there is an nth root of unity $\omega = e^{\frac{2\pi i}{n}}$ such that:

- The nth roots of unity are $1, \omega, \omega^2, \ldots, \omega^{n-1}$
- $1, \omega, \omega^2, \ldots, \omega^{n-1}$ form the vertices of a regular n-gon
- $1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0$

9 The nth roots of any complex number s lie on the vertices of a regular n-gon with its centre at the origin.

10 If z_1 is one root of the equation $z^n = s$, and $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are the nth roots of unity, then the roots of $z^n = s$ are given by $z_1, z_1\omega, z_1\omega^2, \ldots, z_1\omega^{n-1}$.