Summary of key points

3 If $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, then $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j}$

and $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \ddot{\mathbf{r}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j}$

4 $\mathbf{v} = \int \mathbf{a}\,dt$ and $\mathbf{r} = \int \mathbf{v}\,dt$

Variable acceleration 3

You might need to use A-level calculus techniques to model the motion of a particle moving in one dimension with acceleration which varies with time.

You can revise the relationships between displacement, velocity and acceleration for a particle moving in one dimension on page 160.

Worked example

A particle P is moving in a straight line. At time t seconds, $t \geqslant 0$, the velocity of P, $v\,\mathrm{m\,s^{-1}}$, is given by

$$v = \begin{cases} 1 + 2\sin pt, & 0 \leqslant t \leqslant 10\pi \\ \cos 2pt, & 10\pi \leqslant t \leqslant 20\pi \end{cases}$$

where p is a constant. The initial acceleration of P is $0.2\,\mathrm{m\,s^{-2}}$.

(a) Find the value of p. **(3 marks)**

For the motion from $0 \leqslant t \leqslant 10\pi$

$a = \frac{dv}{dt} = 2p\cos pt$

When $t = 0$, $a = 0.2$, so

$0.2 = 2p\cos 0$

$p = 0.1$

(b) Find the displacement of the particle from its starting position after

(i) 10π seconds

(ii) 1 minute. **(8 marks)**

For the motion from $0 \leqslant t \leqslant 10\pi$:

(i) $s = \int v\,dt = \int_0^{10\pi} (1 + 2\sin 0.1t)\,dt$

$= [t - 20\cos 0.1t]_0^{10\pi}$

$= (10\pi - 20\cos\pi) - (0 - 20\cos 0)$

$= 10\pi + 20 - 0 + 20$

So displacement after 10π seconds

$= 10\pi + 40$

$= 71.4159\ldots$ m

(ii) For the motion from $10\pi \leqslant t \leqslant 20\pi$:

$s = \int v\,dt = \int_{10\pi}^{60} (\cos 0.2t)\,dt$

$= [5\sin 0.2t]_{10\pi}^{60}$

$= 5(\sin 12 - \sin 2\pi)$

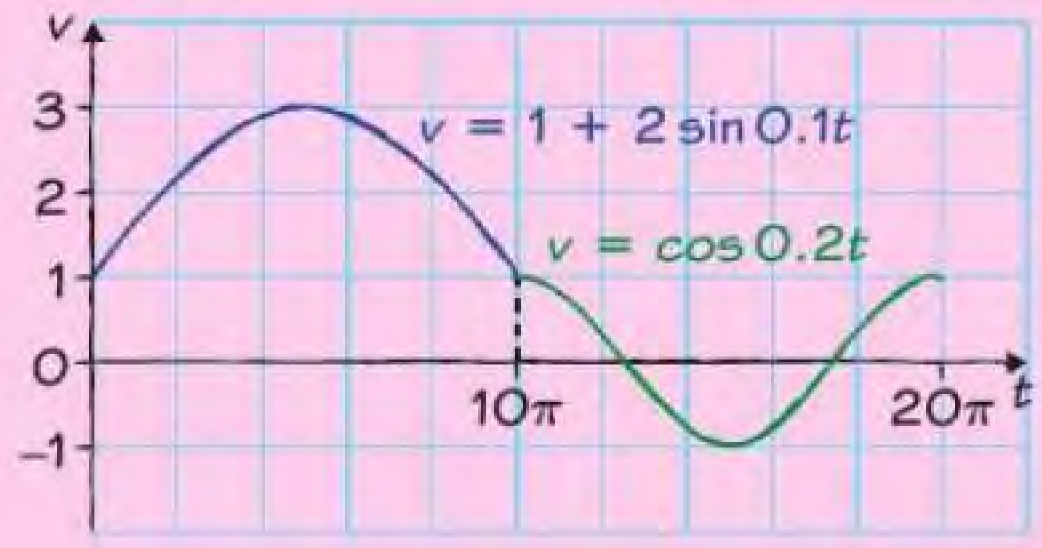
$= -2.6828\ldots$ m

So displacement after 60 seconds is

$71.4159\ldots - 2.6828\ldots = 68.7$ m (3 s.f.)

Problem solved!

This is a **piecewise function**. You use the top part of the function for $0 \leqslant t \leqslant 10\pi$ and the bottom part of the function for $10\pi \leqslant t \leqslant 20\pi$.



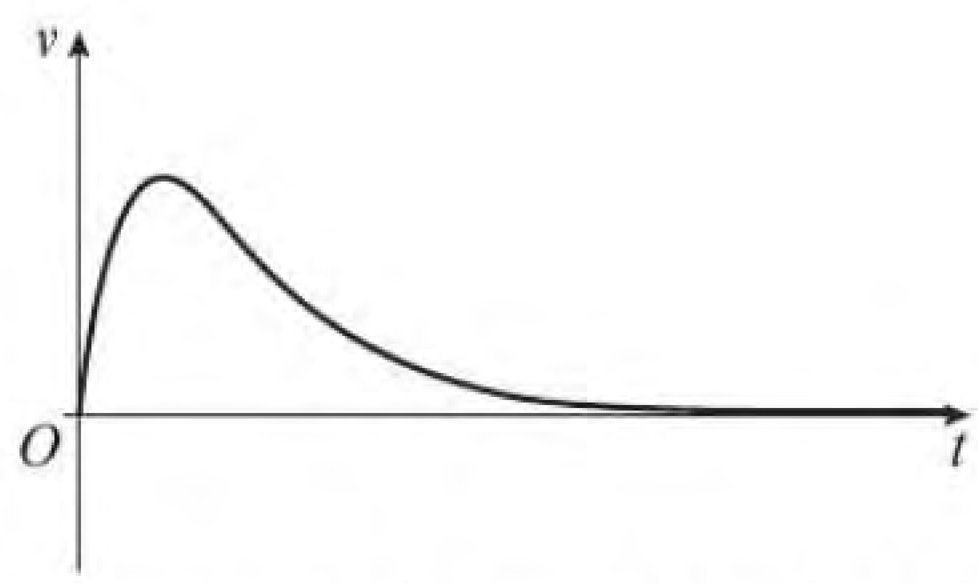
In part (a) you only need to consider the motion of the particle at $t = 0$, so you can just use the top part of the function.

You will need to use problem-solving skills throughout your exam – **be prepared!**

To find acceleration, differentiate the expression for velocity with respect to time. Remember that p is a constant.

Now try this

The diagram shows the velocity–time graph for a particle moving on the positive x-axis.



The velocity of the particle, $v\,\mathrm{m\,s^{-1}}$, at time t seconds is given by

$$v = \frac{t}{e^{3t}},\ t \geqslant 0$$

(a) Find the maximum velocity of the particle. Answer **(5 marks)**

(b) Find the total distance travelled by the particle in the first 2 seconds of its motion, giving your answer correct to 3 significant figures. **(4 marks)**

Answer

In part (a) you will need to use the product rule for differentiation, and in part (b) you will need to use integration by parts.

You can revise these on pages 90 and 104 respectively.

Calculus with vectors

You can use the relationships between displacement, velocity and acceleration to solve problems in two dimensions.

Your vectors should be given in terms of the unit vectors **i** and **j**, and you are always integrating or differentiating with respect to time.

Displacement $= f(t)\mathbf{i} + g(t)\mathbf{j}$

Differentiate ↓ ↑ Integrate

Velocity $= f'(t)\mathbf{i} + g'(t)\mathbf{j}$

Differentiate ↓ ↑ Integrate

Velocity $= f''(t)\mathbf{i} + g''(t)\mathbf{j}$

Worked example

A particle P of mass 0.6 kg is moving in a horizontal plane under the action of a single force **F** N. The velocity of the particle **v** m s^{-1} at time t seconds is given by

$$\mathbf{v} = 80\sqrt{t}\,\mathbf{i} + t^3\mathbf{j},\ t \geqslant 0$$

Find

(a) the acceleration of P when $t = 4$ **(3 marks)**

$$\underset{\sim}{a} = \frac{d\underset{\sim}{v}}{dt} = \frac{40}{\sqrt{t}}\underset{\sim}{i} + 3t^2\underset{\sim}{j}$$

When $t = 4$,

$$\underset{\sim}{a} = \frac{40}{\sqrt{4}}\underset{\sim}{i} + 3(4^2)\underset{\sim}{j} = 20\underset{\sim}{i} + 48\underset{\sim}{j}$$

(b) the magnitude of **F** when $t = 4$. **(3 marks)**

$$F = m\underset{\sim}{a} = 0.6(20\underset{\sim}{i} + 48\underset{\sim}{j})$$
$$= 12\underset{\sim}{i} + 28.8\underset{\sim}{j}$$
$$|F| = \sqrt{12^2 + 28.8^2} = 31.2\text{ N}$$

Golden rules

- ✓ Differentiate or integrate each component of a vector **separately**.
- ✓ Read questions carefully – if you are asked to find **speed** or **distance** you will need to find the magnitude of the velocity or displacement vector.
- ✓ If you are integrating a vector quantity, your **constant of integration** should be a vector.

You need to include a constant of integration, which will be a **vector**. Use the fact that **r** = 12**i** − 3**j** when $t = 3$ to find the constant **c**. It's a good idea to write your final answer giving the **i** component and **j** component as separate functions of t.

Worked example

A particle is moving in a horizontal plane with velocity (3**i** + 10t**j**) m s^{-1}. Given that the particle passes through the point with position vector (12**i** − 3**j**) m at time $t = 3$ seconds, find an expression for the displacement of the particle, **r** m, at time t seconds. **(3 marks)**

$$\underset{\sim}{r} = \int \underset{\sim}{v}\,dt = \int (3\underset{\sim}{i} + 10t\underset{\sim}{j})\,dt$$
$$= 3t\underset{\sim}{i} + 5t^2\underset{\sim}{j} + \underset{\sim}{c}$$

When $t = 3$, $\underset{\sim}{r} = 12\underset{\sim}{i} - 3\underset{\sim}{j}$, so

$$9\underset{\sim}{i} + 45\underset{\sim}{j} + \underset{\sim}{c} = 12\underset{\sim}{i} - 3\underset{\sim}{j}$$
$$\underset{\sim}{c} = 3\underset{\sim}{i} - 48\underset{\sim}{j}$$

So $\underset{\sim}{r} = (3t + 3)\underset{\sim}{i} + (5t^2 - 48)\underset{\sim}{j}$

Now try this

A particle Q is moving in a horizontal plane. The acceleration of the particle, **a** m s^{-2} at time t seconds is given by

$$\mathbf{a} = (4t - 1)\mathbf{i} - \mathbf{j},\ t \geqslant 0$$

When $t = 0$, the particle is at the origin and moving with velocity −3**i** m s^{-1}.

(a) Find the velocity of the particle when $t = 2$. Answer **(4 marks)**

(b) Find the time at which the particle is moving parallel to the unit vector **j**. Answer **(3 marks)**

(c) Find the distance of the particle from the origin when $t = 2$. **(6 marks)** Answer

Watch out! You need to integrate all of $(4 - 2t)$ and use the initial condition to find the **i** component of the velocity. The particle will be moving parallel to the vector **j** when this component is equal to zero.