## Summary of key points

1. The **horizontal** motion of a projectile is modelled as having **constant velocity** ($a = 0$). You can use the formula $s = vt$.

2. The **vertical** motion of a projectile is modelled as having **constant acceleration** due to gravity ($a = g$).

3. When a particle is projected with initial velocity $U$, at an angle $\alpha$ above the horizontal:
   - The **horizontal component** of the initial velocity is $U\cos\alpha$
   - The **vertical component** of the initial velocity is $U\sin\alpha$

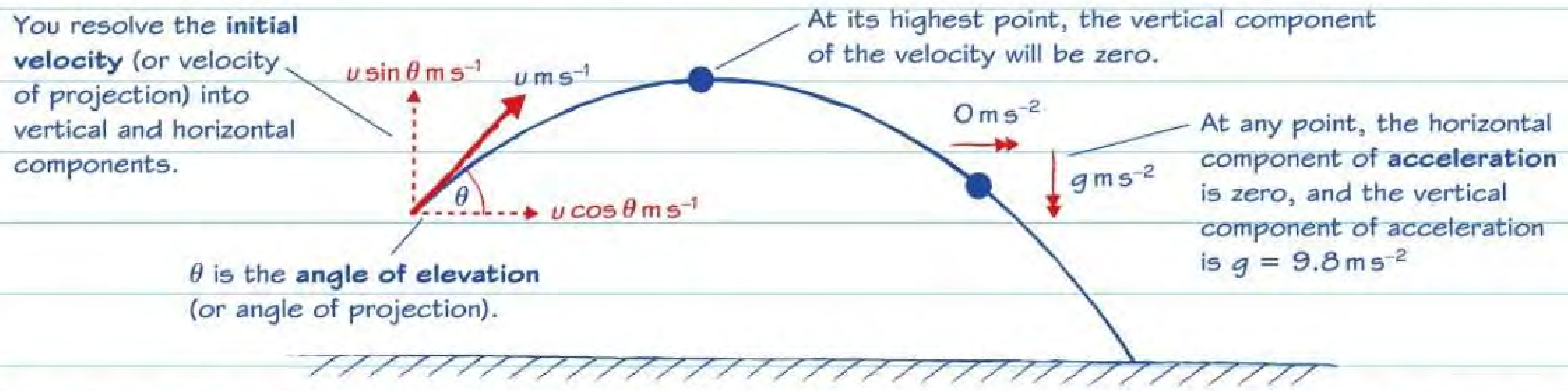
4. A projectile reaches its point of greatest height when the vertical component of its velocity is equal to 0.

5. For a particle which is projected from a point on a horizontal plane with an initial velocity $U$ at an angle $\alpha$ above the horizontal, and that moves freely under gravity:
   - Time of flight $= \dfrac{2U\sin\alpha}{g}$
   - Time to reach greatest height $= \dfrac{U\sin\alpha}{g}$
   - Range on horizontal plane $= \dfrac{U^2\sin 2\alpha}{g}$
   - Equation of trajectory: $y = x\tan\alpha - gx^2\dfrac{(1+\tan^2\alpha)}{2U^2}$

   where $y$ is the vertical height of the particle, $x$ is the horizontal distance from the point of projection, and $g$ is the acceleration due to gravity.

# Projectiles

You can model a projectile as a particle moving freely under gravity. This means that you ignore air resistance, and any rotational forces acting on the particle.




## Worked example

A particle is projected from a point $O$ on a horizontal plane with speed 15 m s$^{-1}$, at an angle of $\theta$ above the horizontal, where $\tan\theta = \frac{4}{3}$

Find

(a) the time taken for the particle to hit the plane **(4 marks)**

R(↑): $u_y = 15 \sin\theta$

$= 12$ m s$^{-1}$

$s = 0$, $u = 12$, $v = ?$, $a = -9.8$, $t = ?$

$s = ut + \frac{1}{2}at^2$

$0 = 12t - 4.9t^2$

$t = 0$ or $2.448\ldots$

The particle hits the plane after 2.4 s (2 s.f.)

(b) the horizontal distance travelled by the particle. **(2 marks)**

R(→): $u_x = 15 \cos\theta$

$= 9$ m s$^{-1}$

$s = vt$

$= 9 \times 2.448\ldots = 22$ m (2 s.f.)

## Problem solved!

In projectiles questions you usually start by analysing the **vertical motion** of the particle. The vertical motion has **constant acceleration** so you can use the *suvat* formulae. Revise these on pages 150 and 151. The horizontal motion has constant speed (zero acceleration) so you can use $s = vt$ (or distance = speed × time).

You will need to use problem-solving skills throughout your exam – **be prepared!**

When resolving the **initial velocity** into components you can use $u_x$ to represent the horizontal component and $u_y$ to represent the vertical component.

In part (a), you could also use symmetry to deduce that the vertical component of the velocity will be −12 when the particle hits the plane:

$v = u + at$

$-12 = 12 - 9.8t$

$t = 2.4$ (2 s.f.)

## Now try this

**1** A stone is projected at an angle of elevation of 40° from the top of a building, 28 m above level ground. The stone strikes the ground 6 seconds later. By modelling the stone as a particle moving freely under gravity, find

(a) the speed of projection **(5 marks)** Answer

(b) the distance from the base of the building to the point where the stone strikes the ground. Answer **(2 marks)**

**2** A particle is projected horizontally from a height of 10 m above a level plane, at a speed of 18 m s$^{-1}$. Find the speed of the particle at the point where it hits the plane. **(6 marks)** Answer

Watch out! Speed is the **magnitude** of the velocity. You need to consider both the vertical and horizontal components.

Had a look ☐ Nearly there ☐ Nailed it! ☐

# Projectile formulae

You might be asked to derive general formulae associated with projectile motion in your exam.

## Worked example

A particle is projected from a point $O$ on level ground at a speed of $7\,\text{m s}^{-1}$ and an angle of $\alpha$ above the horizontal. When the particle has moved a horizontal distance $x$ m, its height above the point of projection is $y$ m.

(a) Show that $y = x\tan\alpha - \frac{1}{10}x^2(1 + \tan^2\alpha)$. **(7 marks)**

R(↑): $u_y = 7\sin\alpha$

R(→): $u_x = 7\cos\alpha$

Using $s = ut + \frac{1}{2}at^2$ for the vertical motion:

$y = 7\sin\alpha t - 4.9t^2$ ①

Using $s = vt$ for the horizontal motion:

$x = 7\cos\alpha t$ ②

Rearranging ②: $t = \dfrac{x}{7\cos\alpha}$

Substituting into ①:

$$y = 7\sin\alpha\left(\frac{x}{7\cos\alpha}\right) - 4.9\left(\frac{x}{7\cos\alpha}\right)^2$$

$$= x\tan\alpha - \tfrac{1}{10}x^2\sec^2\alpha$$

$$= x\tan\alpha - \tfrac{1}{10}x^2(1 + \tan^2\alpha)$$

(b) Given that when $x = 3$, $y = 1.5$, find two possible values for $\alpha$. **(5 marks)**

$1.5 = 3\tan\alpha - 0.9(1 + \tan^2\alpha)$

Let $u = \tan\alpha$:

$1.5 = 3u - 0.9(1 + u^2)$

$3u^2 - 10u + 8 = 0$

$(3u - 4)(u - 2) = 0$

$u = 2$ or $\frac{4}{3}$, so $\tan\alpha = 2$ or $\frac{4}{3}$

So $\alpha = 63.4°$ or $53.1°$ (3 s.f.)

## Problem solved!

You can write expressions for $y$ and $x$ in terms of $t$ and $\alpha$. These are **parametric equations** for the path of the projectile, with parameter $t$. You can convert them into cartesian equations by eliminating $t$.

Revise parametric equations on page 86.

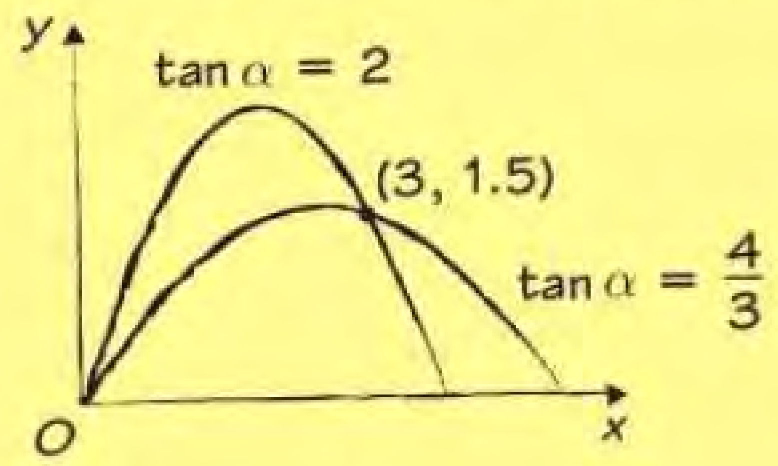
You will need to use problem-solving skills throughout your exam – **be prepared!**

Use trigonometric identities to get your equation in the required form.

$\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ $\quad \sec\theta \equiv \dfrac{1}{\cos\theta}$

$\sec^2\theta \equiv 1 + \tan^2\theta$

Substitute $x = 3$ and $y = 1.5$ into the equation from part (a). This is a quadratic equation in $\tan\alpha$. There are two possible values of $\tan\alpha$, which correspond to the two possible paths which pass through the point (3, 1.5).




### Parabolic path

The example on the left is one particular case of the general formula for the path of a projectile. If a particle is projected from the origin with initial velocity $U$ and angle of elevation $\alpha$, the equation of its trajectory will be:

$$y = x\tan\alpha - gx^2\left(\frac{1 + \tan^2\alpha}{2U^2}\right)$$

This is a quadratic curve (or **parabola**).

## Now try this

A particle is projected from a point $O$ on level ground at a speed of $U\,\text{m s}^{-1}$ and an angle of $\alpha$ above the horizontal. Show that

(a) the time of flight of the particle is given by $\dfrac{2U\sin\alpha}{g}$ Answer **(4 marks)**

(b) the horizontal range of the particle is given by $\dfrac{U^2\sin 2\alpha}{g}$ **(4 marks)** Answer

(a) Use $s = ut + \frac{1}{2}at^2$, leaving your answer in terms of $g$.

(b) Use your answer from part (a) and the identity $2\sin\theta\cos\theta \equiv \sin 2\theta$