Summary of key points

1 • 2π radians = 360° • π radians = 180° • 1 radian = $\frac{180°}{\pi}$

2 • $30° = \frac{\pi}{6}$ radians • $45° = \frac{\pi}{4}$ radians • $60° = \frac{\pi}{3}$ radians

• $90° = \frac{\pi}{2}$ radians • $180° = \pi$ radians • $360° = 2\pi$ radians

3 You need to learn the exact values of the trigonometric ratios of these angles measured in radians.

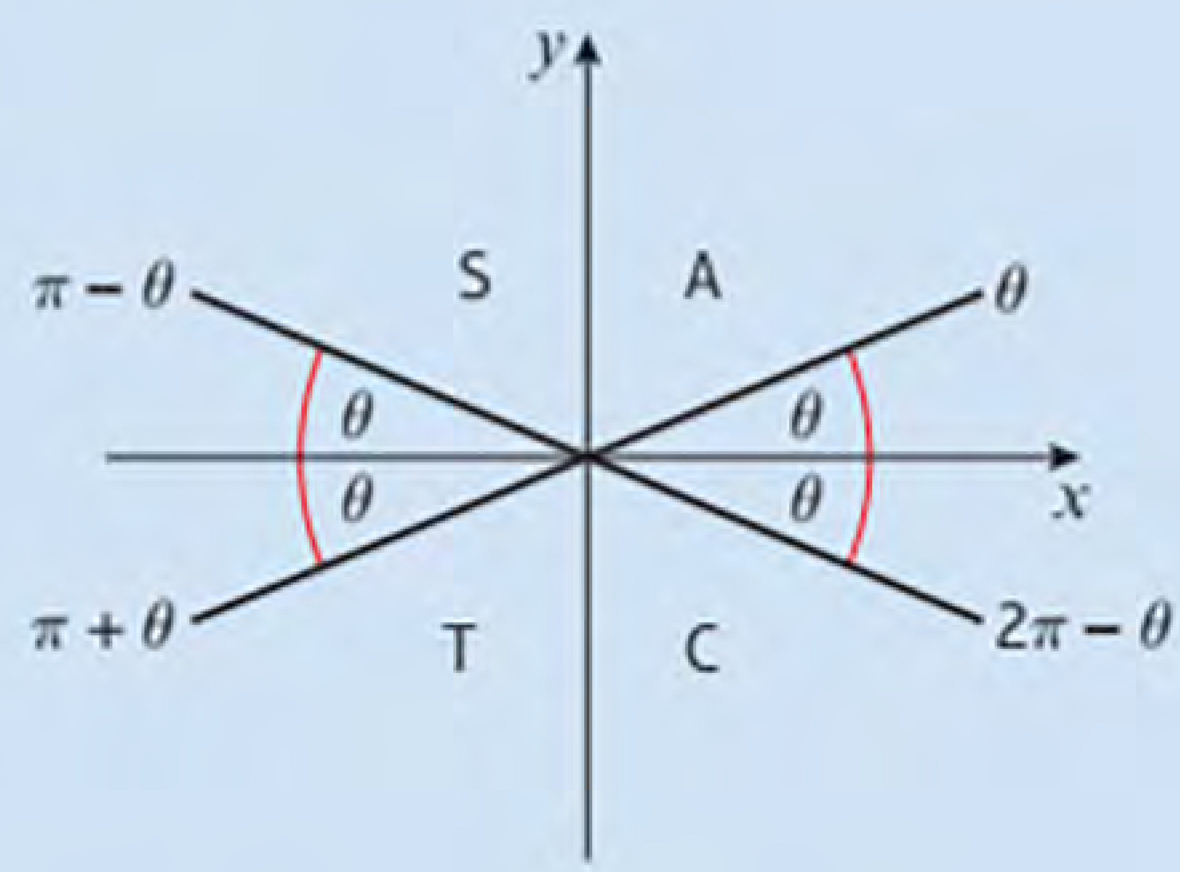
• $\sin\frac{\pi}{6} = \frac{1}{2}$ • $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ • $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

• $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ • $\cos\frac{\pi}{3} = \frac{1}{2}$ • $\tan\frac{\pi}{3} = \sqrt{3}$

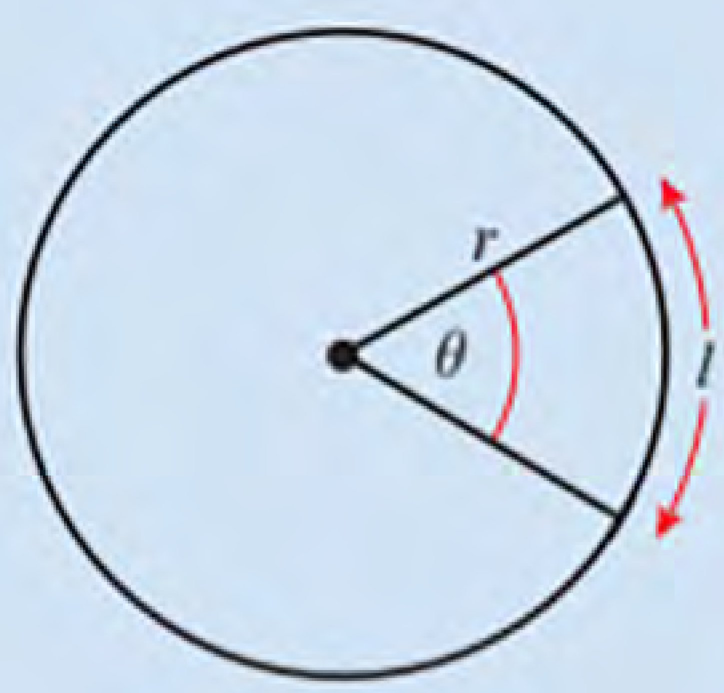
• $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ • $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ • $\tan\frac{\pi}{4} = 1$

4 You can use these rules to find sin, cos or tan of any positive or negative angle measured in radians using the corresponding acute angle made with the x-axis, θ.

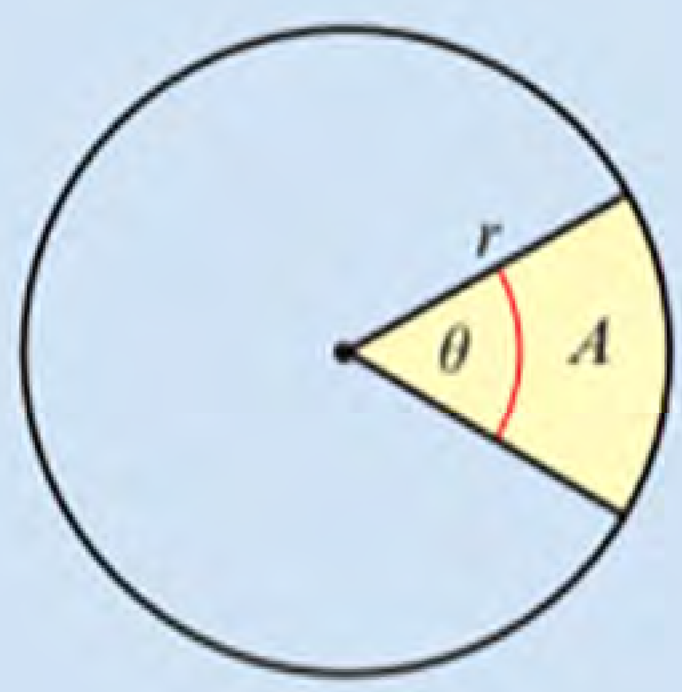
- $\sin(\pi - \theta) = \sin\theta$
- $\sin(\pi + \theta) = -\sin\theta$
- $\sin(2\pi - \theta) = -\sin\theta$
- $\cos(\pi - \theta) = -\cos\theta$
- $\cos(\pi + \theta) = -\cos\theta$
- $\cos(2\pi - \theta) = \cos\theta$
- $\tan(\pi - \theta) = -\tan\theta$
- $\tan(\pi + \theta) = \tan\theta$
- $\tan(2\pi - \theta) = -\tan\theta$



5 To find the arc length l of a sector of a circle use the formula $l = r\theta$, where r is the radius of the circle and θ is the angle, in radians, contained by the sector.



6 To find the area A of a sector of a circle use the formula $A = \frac{1}{2}r^2\theta$, where r is the radius of the circle and θ is the angle, in radians, contained by the sector.



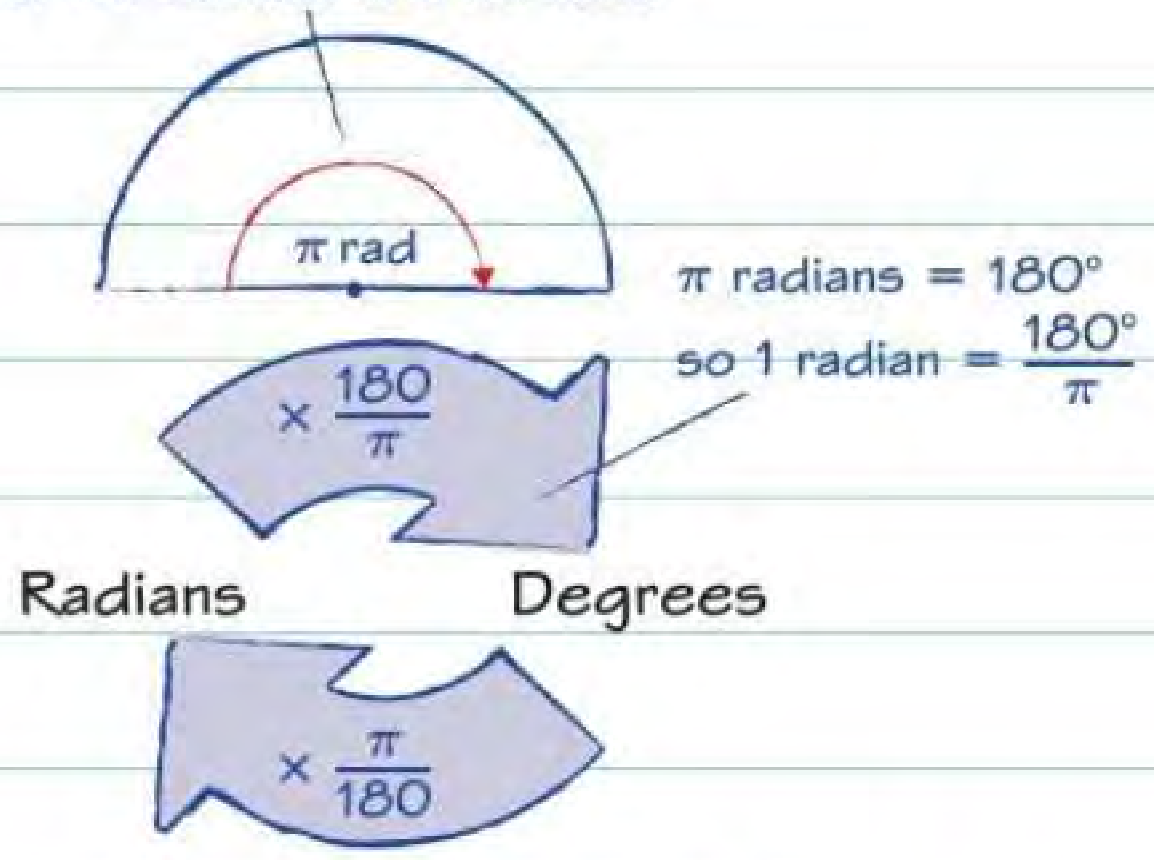
7 The area of a segment in a circle of radius r is

$$A = \frac{1}{2}r^2(\theta - \sin\theta)$$

Radians, arcs and sectors

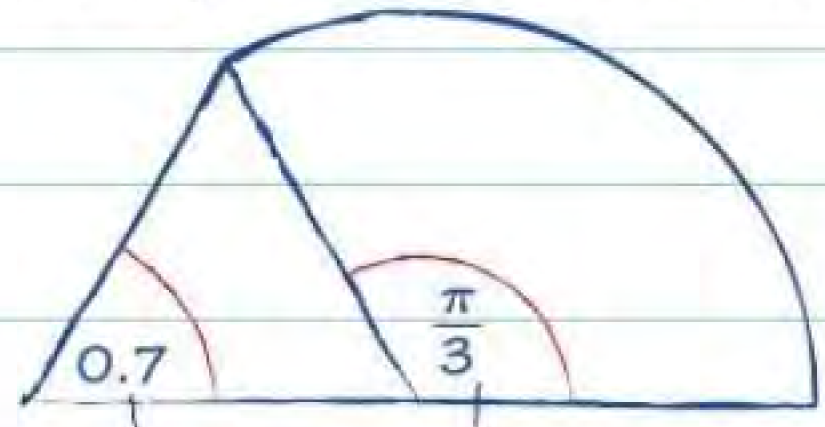
If you have to solve **length** or **area** problems involving circles and triangles in your A-level exam, any angles will usually be measured in **radians**.

There are π radians in a semicircle, or 2π radians in a full turn.



Writing radians

Make sure you know whether you are working in **radians** or **degrees**. These two angles are measured in radians.

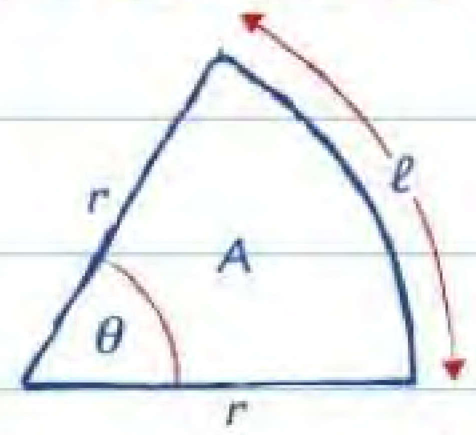


If the angle is written in terms of π, then it is definitely in radians.

This means 0.7 radians. This angle could also be written as 0.7 rad.

Radians in length and area formulae

Radians are really useful for working out **arc lengths** and **sector areas**. Here are two formulae that **only work** if the angle θ is measured in **radians**:



1 Arc length $\ell = r\theta$

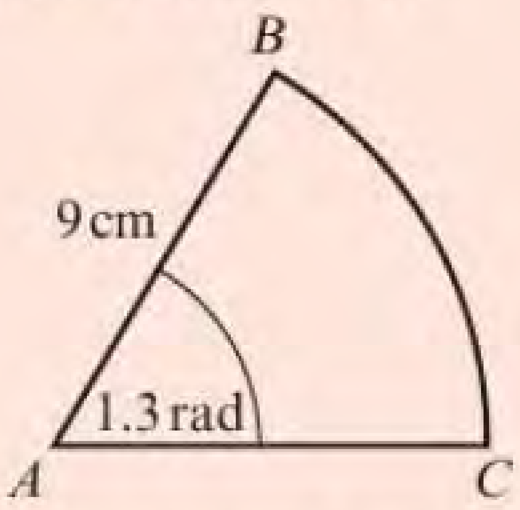
This is the Greek letter θ. It is sometimes used to represent an unknown **angle**.

2 Sector area $A = \frac{1}{2}r^2\theta$

Neither of these formulae is in the booklet, so make sure you **learn** them.

Worked example

The diagram shows ABC, a sector of a circle with centre A and radius 9 cm.



Given that the size of $\angle BAC$ is 1.3 radians, find

(a) the length of the arc BC **(2 marks)**

$\ell = r\theta = 9 \times 1.3 = 11.7$ cm

(b) the area of the sector ABC. **(2 marks)**

$A = \frac{1}{2}r^2\theta = \frac{1}{2} \times 9^2 \times 1.3 = 52.65$ cm^2

The angle, θ, is given in radians, so you can use the formulae for arc length and sector area given above.

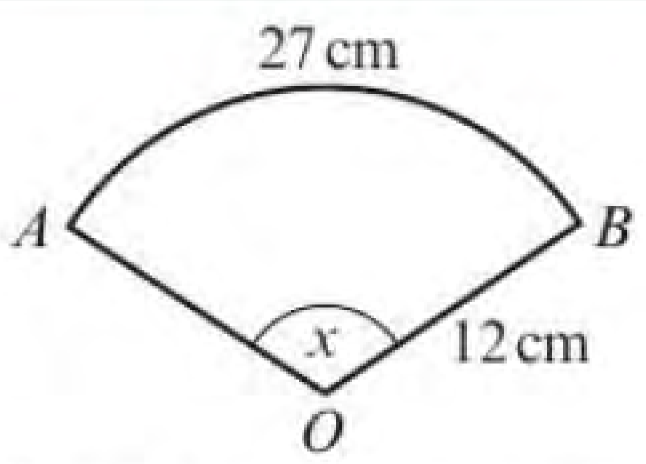
Remember to **write out the formula** first. That way, you will get some credit even if you make a mistake when you're substituting your values.

Now try this

The diagram shows a sector OAB of a circle with radius 12 cm and centre O. The length of the arc AB is 27 cm. Find

(a) the size of $\angle AOB$ in radians Answer **(2 marks)**

(b) the area of the sector. Answer **(2 marks)**

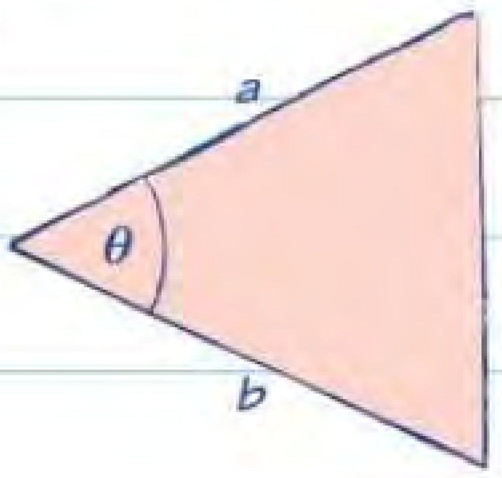


Areas of triangles

You can use this formula to find the area of **any** triangle if you know two sides and the angle between them (SAS):

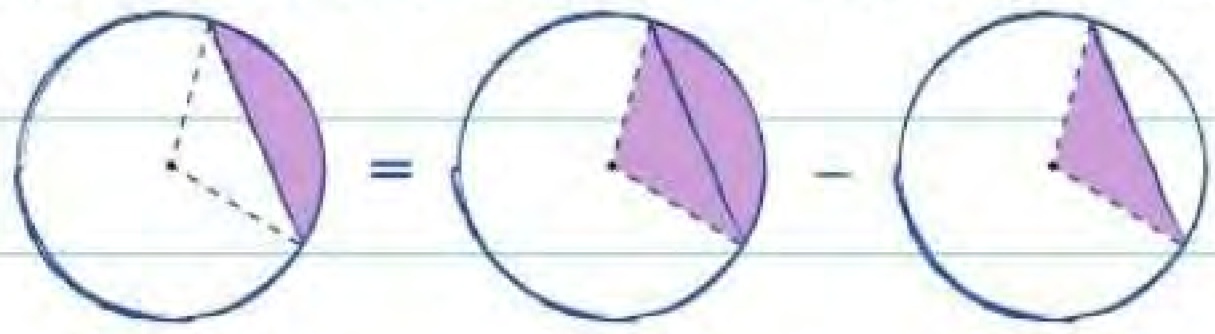
Area $= \frac{1}{2}ab\sin\theta$

This formula is not in the booklet so you need to **learn** it.



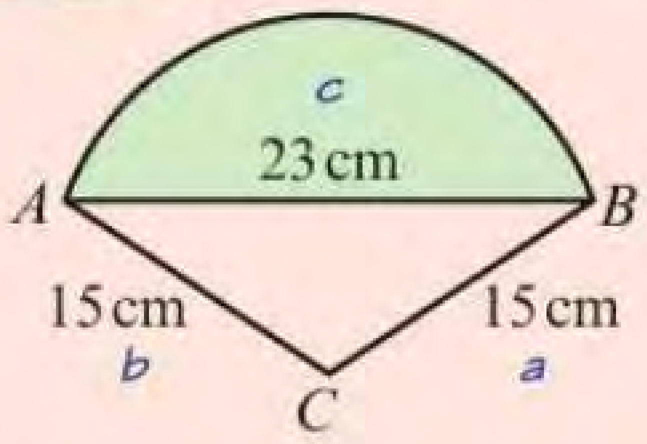
Areas of segments

A chord divides a circle into two **segments.**



Area of segment = Area of whole sector − Area of triangle

Worked example



Everything in blue is part of the answer.

In the diagram *CAB* is a sector of a circle with centre *C* and radius 15 cm. The chord *AB* is 23 cm long.

(a) Find the size of $\angle ACB$, giving your answer in radians to 3 decimal places. **(3 marks)**

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{15^2 + 15^2 - 23^2}{2 \times 15 \times 15}$$
$$= -0.1755...$$
$$C = \cos^{-1}(-0.1755...) = 1.7472...$$
$$= 1.747 \text{ radians (3 d.p.)}$$

(b) Calculate the area of the sector *CAB*, in cm², correct to 1 decimal place. **(2 marks)**

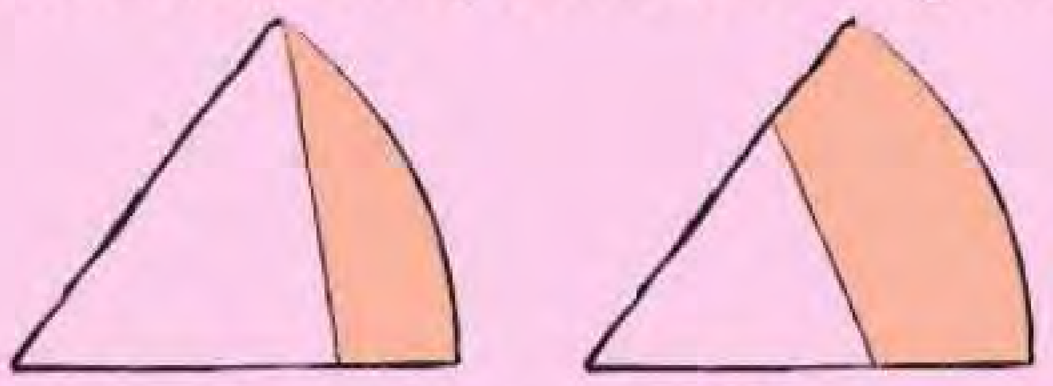
$$A = \frac{1}{2}r^2\theta = \frac{1}{2} \times 15^2 \times 1.7472...$$
$$= 196.56... = 196.6 \text{ cm}^2 \text{ (1 d.p.)}$$

(c) Hence calculate the shaded area, in cm², correct to 1 decimal place. **(3 marks)**

$$\text{Area of } \angle ABC = \frac{1}{2}ab\sin\theta$$
$$= \frac{1}{2} \times 15 \times 15 \times \sin(1.7472...)$$
$$= 110.75... = 110.8 \text{ cm}^2 \text{ (1 d.p.)}$$
$$\text{Shaded area} = 196.56... - 110.75...$$
$$= 85.8 \text{ cm}^2 \text{ (1 d.p.)}$$

Problem solved!

Don't try and learn a single formula for the area of a segment. You might get caught out with areas like these, which aren't segments:



It's much safer and more useful to remember **how to work it out** using the formulae for the **area of a sector** and the **area of a triangle**.

You will need to use problem-solving skills throughout your exam – **be prepared!**

For a reminder about using the **cosine rule** to find a missing angle, look at page 27. To find the area of the shaded **segment**, subtract the area of the triangle from the area of the sector. You can revise areas of sectors on page 76.

BC and *AD* are parallel, so $\angle BCA = \angle CAD$. For part (b), work out $\angle ABC$ using the **sine rule**, then use the fact that the angles in a triangle add up to π radians to find $\angle BAC$. For a reminder about using the sine rule, look at page 28.

Now try this

The diagram shows the cross-section of a tent. The lines *BC* and *AD* are parallel. *ACD* is a sector of a circle with centre *A* and radius 2.08 m. Find

Answer

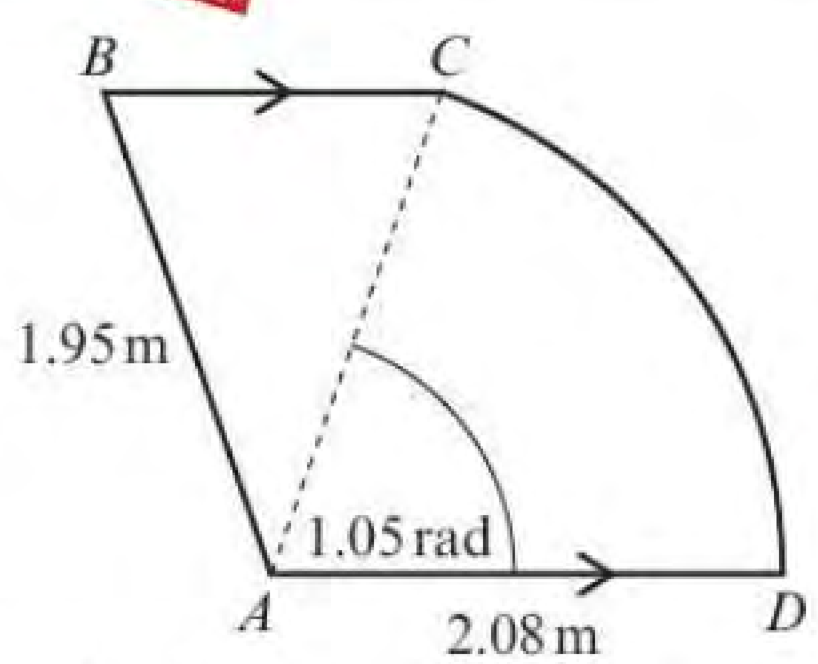
(a) the area of the sector *ACD* in m² to 2 decimal places **(2 marks)**

(b) the size of $\angle BAC$ in radians, to 2 decimal places **(3 marks)**

(c) the area of the entire cross-section *ABCD* of the tent, in m², to 2 decimal places. **(3 marks)**

Answer

Answer



Summary of key points

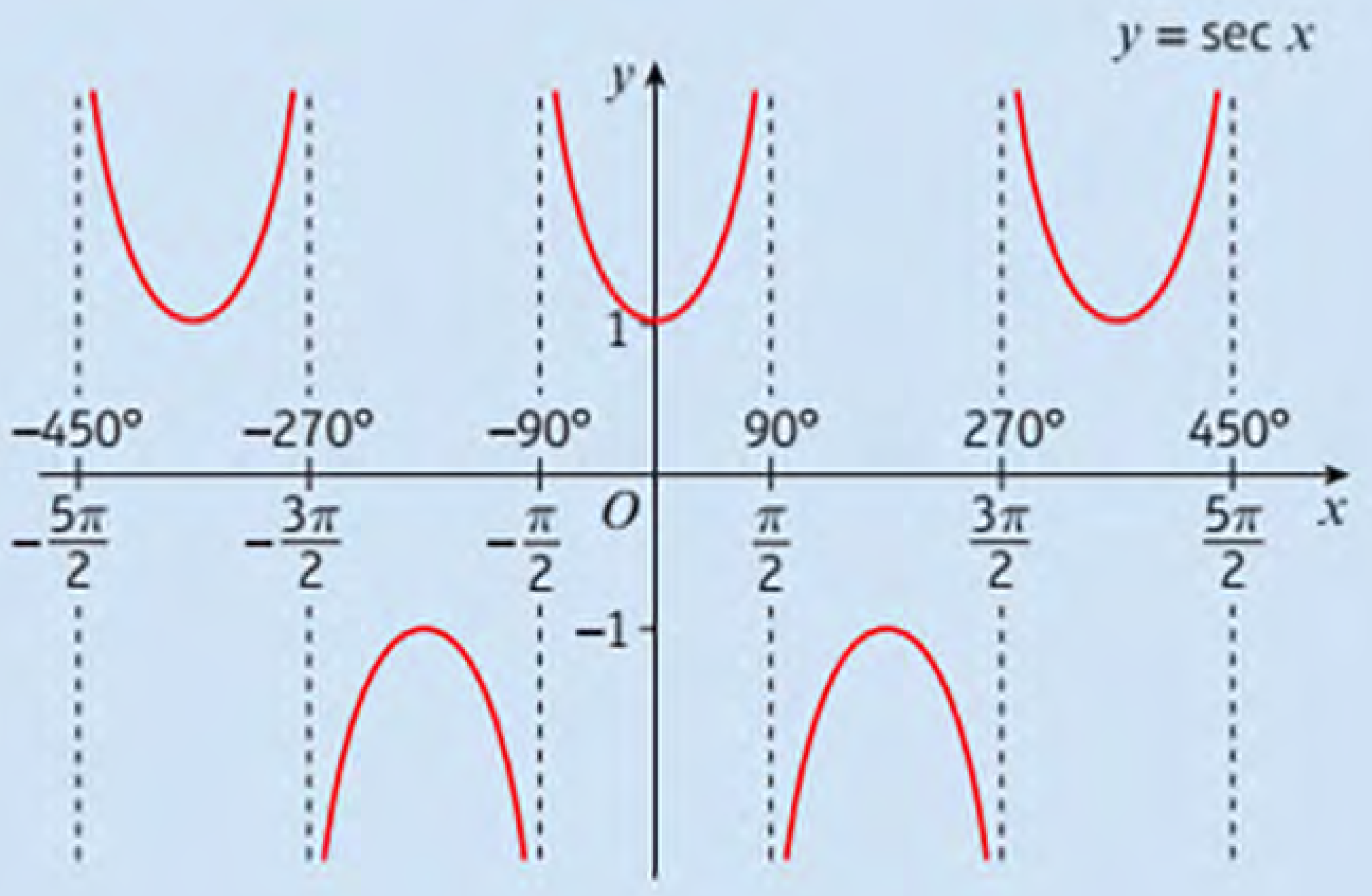
1 • $\sec x = \dfrac{1}{\cos x}$ (undefined for values of x for which $\cos x = 0$)

• $\operatorname{cosec} x = \dfrac{1}{\sin x}$ (undefined for values of x for which $\sin x = 0$)

• $\cot x = \dfrac{1}{\tan x}$ (undefined for values of x for which $\tan x = 0$)

• $\cot x = \dfrac{\cos x}{\sin x}$

2 The graph of $\boldsymbol{y = \sec x}$, $x \in \mathbb{R}$, has symmetry in the y-axis and has period 360° or 2π radians. It has vertical asymptotes at all the values of x for which $\cos x = 0$.

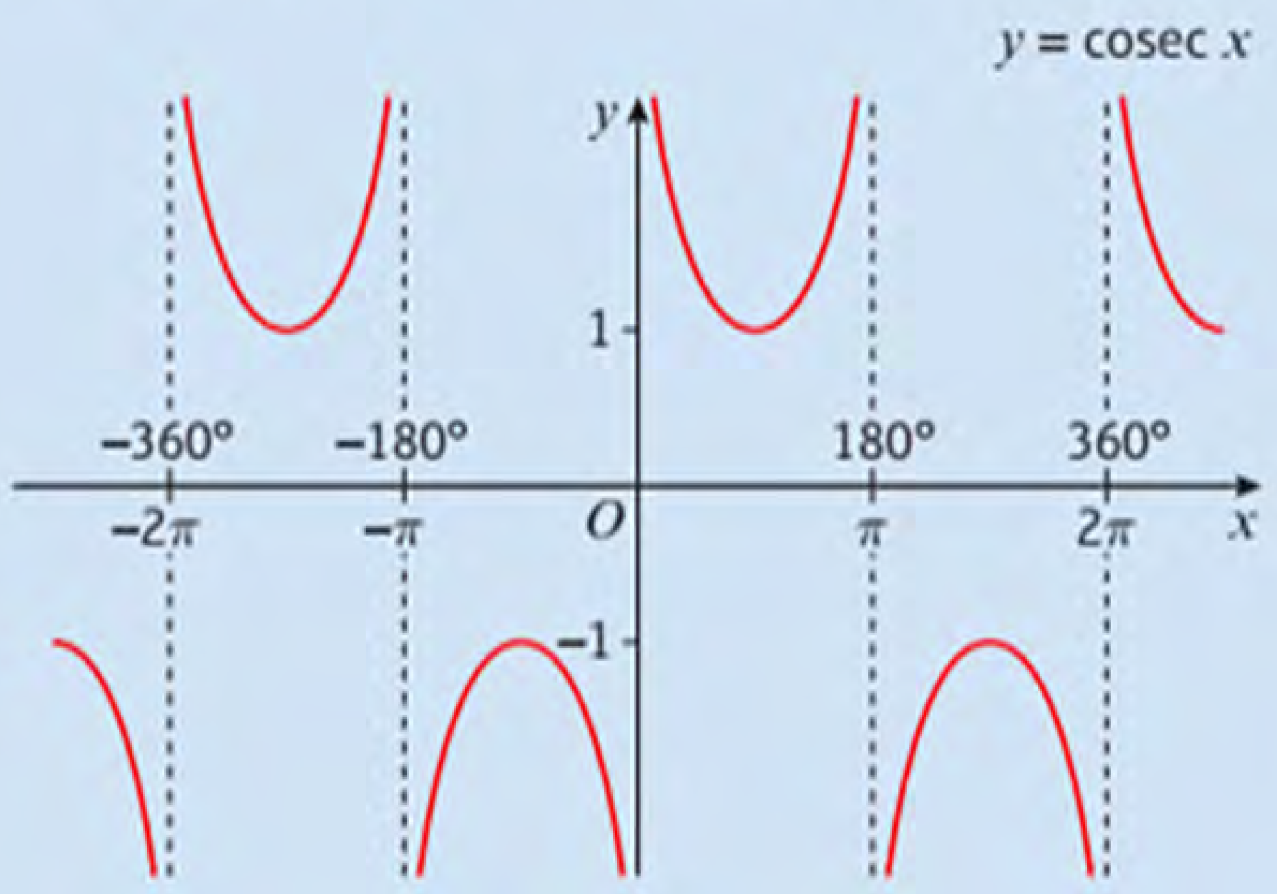


• The domain of $y = \sec x$ is $x \in \mathbb{R}$, $x \neq 90°, 270°, 450°, \ldots$ or any odd multiple of 90°.

• In radians the domain is $x \in \mathbb{R}$, $x \neq \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ or any odd multiple of $\frac{\pi}{2}$

• The range of $y = \sec x$ is $y \leqslant -1$ or $y \geqslant 1$.

3 The graph of $\boldsymbol{y = \operatorname{cosec} x}$, $x \in \mathbb{R}$, has period 360° or 2π radians. It has vertical asymptotes at all the values of x for which $\sin x = 0$.

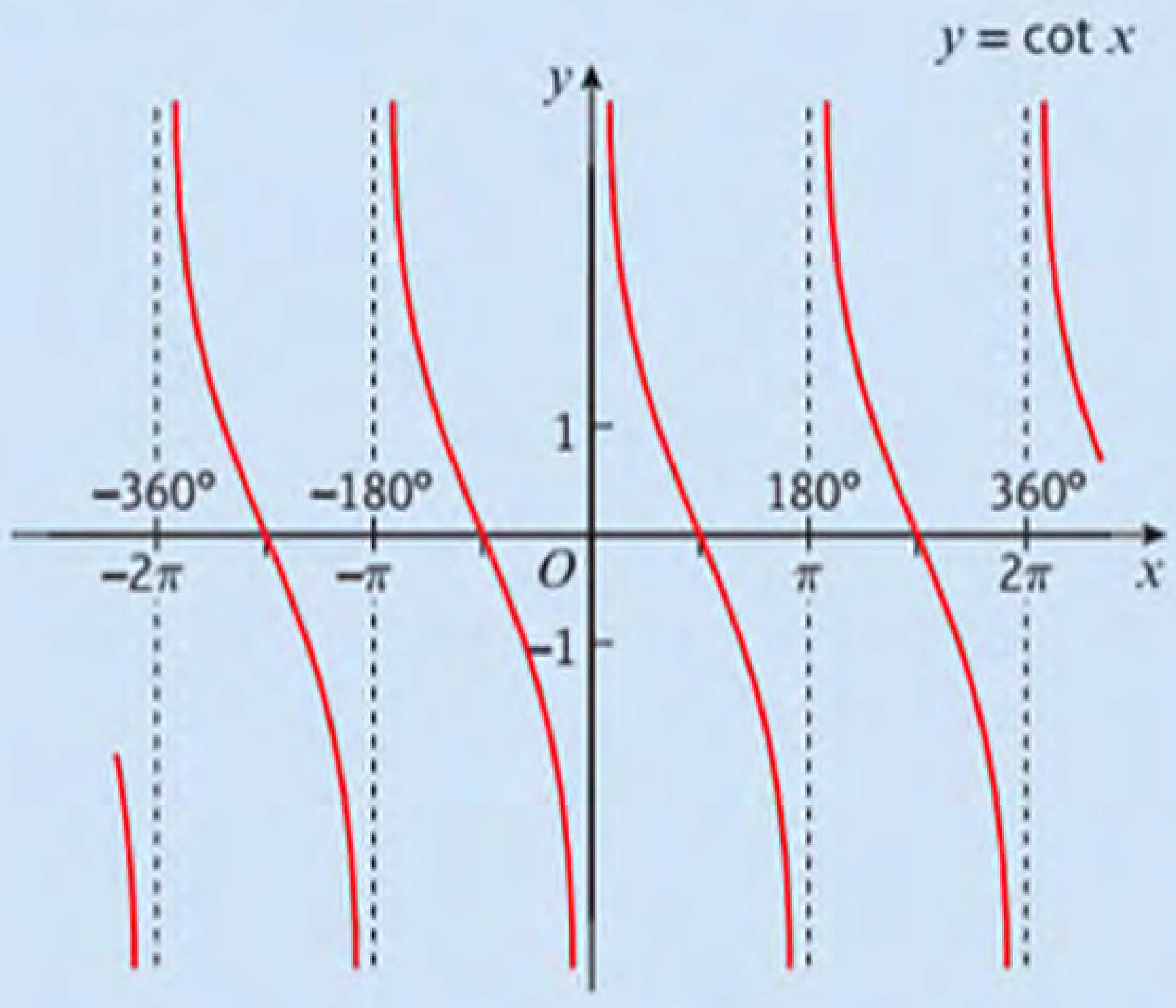


• The domain of $y = \operatorname{cosec} x$ is $x \in \mathbb{R}$, $x \neq 0°, 180°, 360°, \ldots$ or any multiple of 180°.

• In radians the domain is $x \in \mathbb{R}$, $x \neq 0, \pi, 2\pi, \ldots$ or any multiple of π

• The range of $y = \operatorname{cosec} x$ is $y \leqslant -1$ or $y \geqslant 1$.

4 The graph of $\boldsymbol{y = \cot x}$, $x \in \mathbb{R}$, has period 180° or π radians. It has vertical asymptotes at all the values of x for which $\tan x = 0$.



• The domain of $y = \cot x$ is $x \in \mathbb{R}$, $x \neq 0°, 180°, 360°, \ldots$ or any multiple of 180°.

• In radians the domain is $x \in \mathbb{R}$, $x \neq 0, \pi, 2\pi, \ldots$ or any multiple of π.

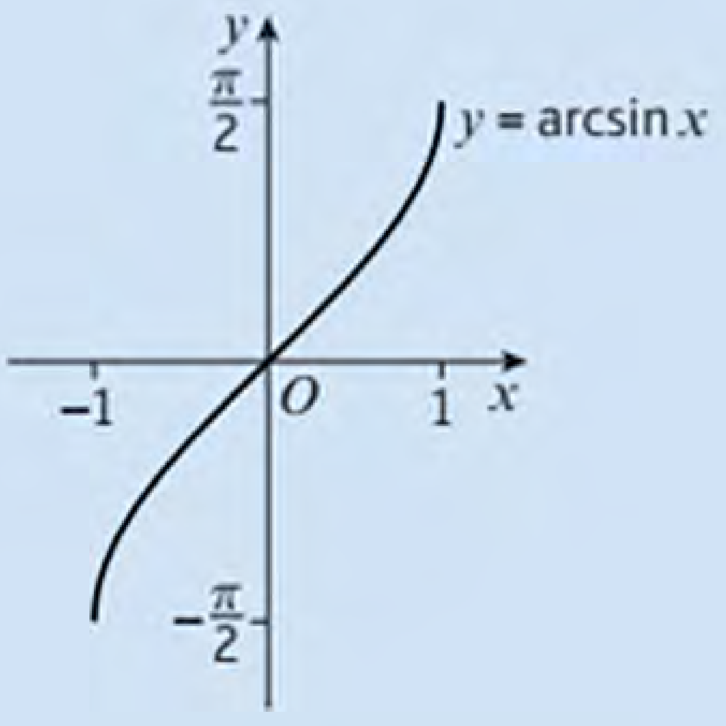
• The range of $y = \cot x$ is $y \in \mathbb{R}$.

5 $\sec x = k$ and $\operatorname{cosec} x = k$ have no solutions for $-1 < k < 1$.

7 The inverse function of $\sin x$ is called **arcsin** $\boldsymbol{x}$.

• The domain of $y = \arcsin x$ is $-1 \leqslant x \leqslant 1$

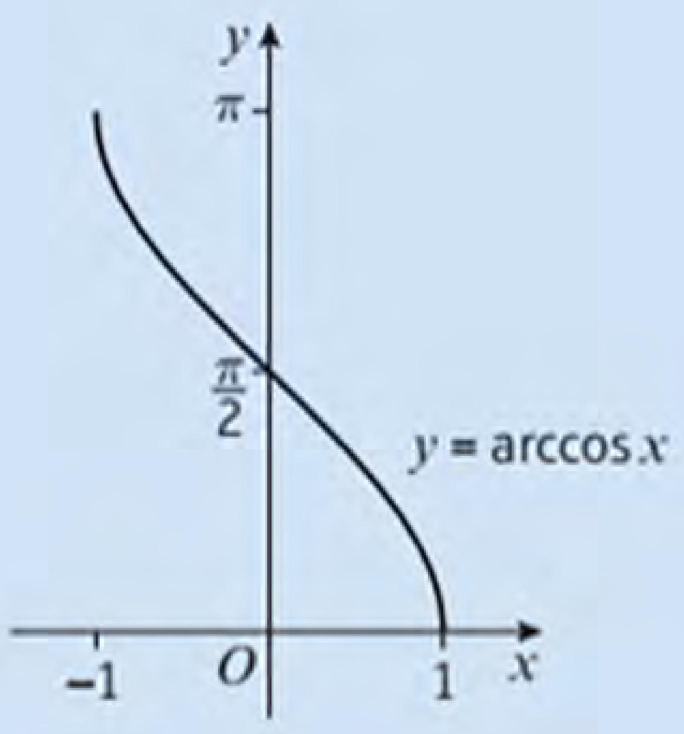
• The range of $y = \arcsin x$ is $-\dfrac{\pi}{2} \leqslant \arcsin x \leqslant \dfrac{\pi}{2}$ or $-90° \leqslant \arcsin x \leqslant 90°$



8 The inverse function of $\cos x$ is called **arccos** $\boldsymbol{x}$.

• The domain of $y = \arccos x$ is $-1 \leqslant x \leqslant 1$

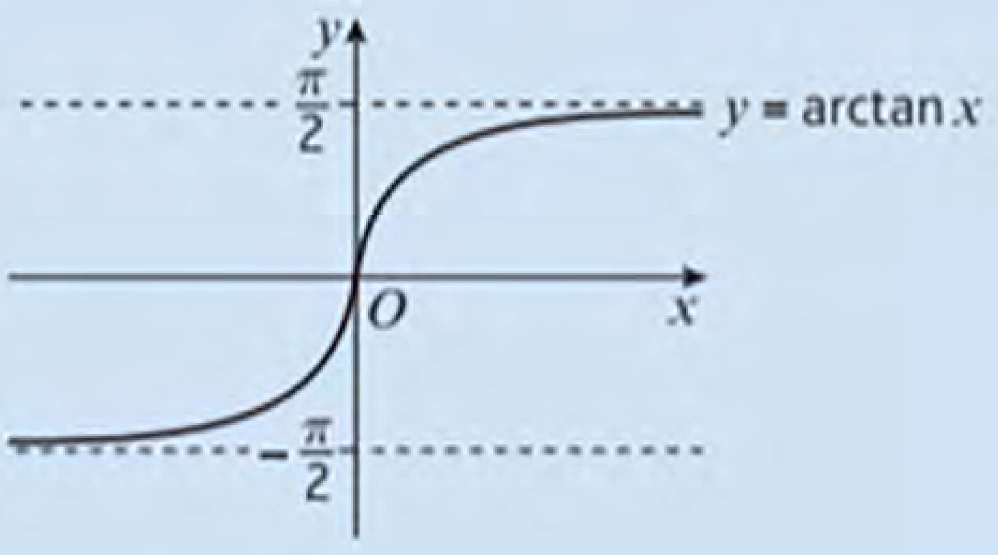
• The range of $y = \arccos x$ is $0 \leqslant \arccos x \leqslant \pi$ or $0° \leqslant \arccos x \leqslant 180°$



9 The inverse function of $\tan x$ is called **arctan** $\boldsymbol{x}$.

• The domain of $y = \arctan x$ is $x \in \mathbb{R}$

• The range of $y = \arctan x$ is $-\dfrac{\pi}{2} < \arctan x < \dfrac{\pi}{2}$ or $-90° < \arctan x < 90°$



Sec, cosec and cot

The **reciprocals** of sin, cos and tan have special names. You need to learn the names of these functions and be able to recognise and sketch their graphs:

$\operatorname{cosec}\theta \equiv \dfrac{1}{\sin\theta}$ $\quad \sec\theta \equiv \dfrac{1}{\cos\theta}$ $\quad \cot\theta \equiv \dfrac{1}{\tan\theta}$

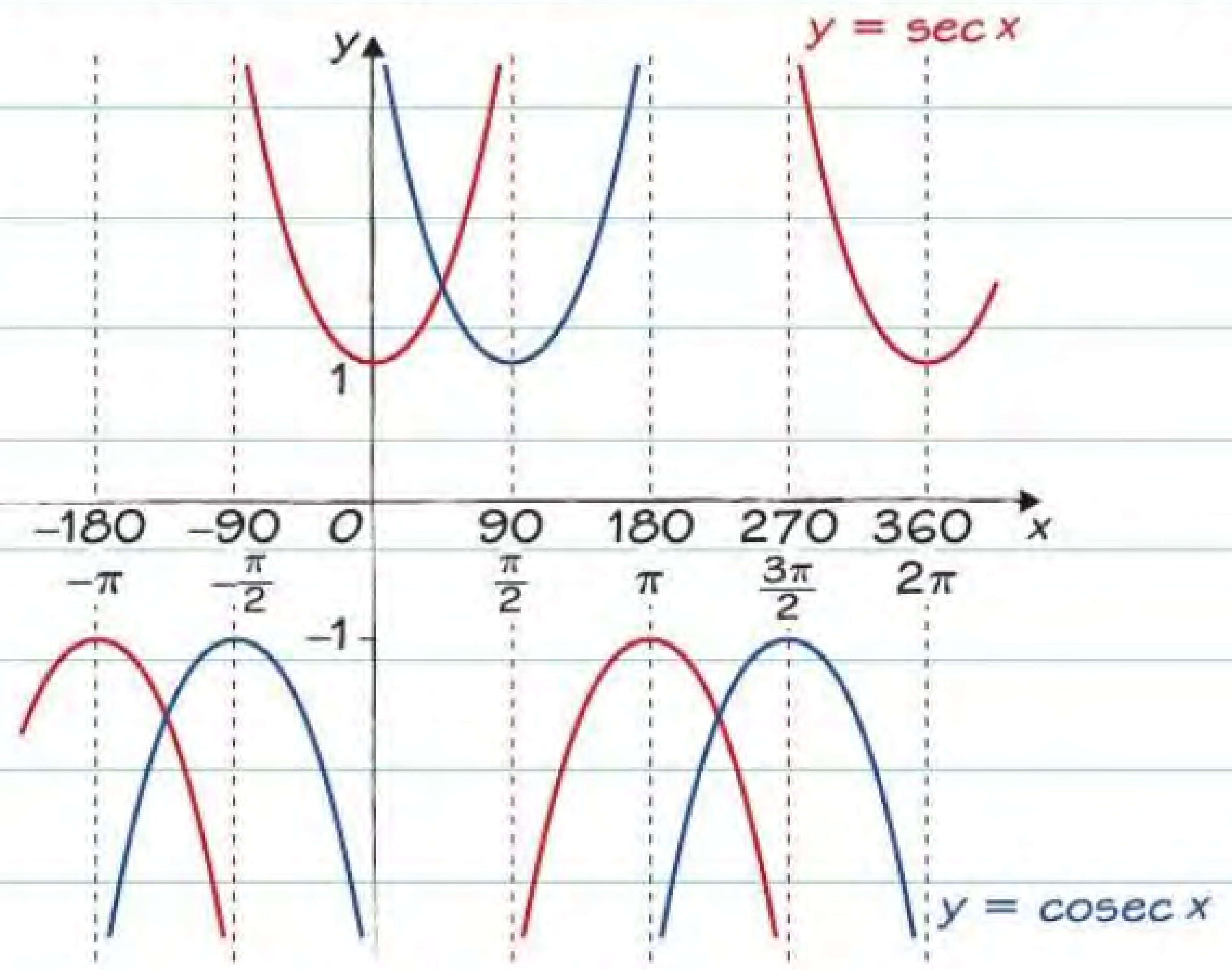
Golden rule

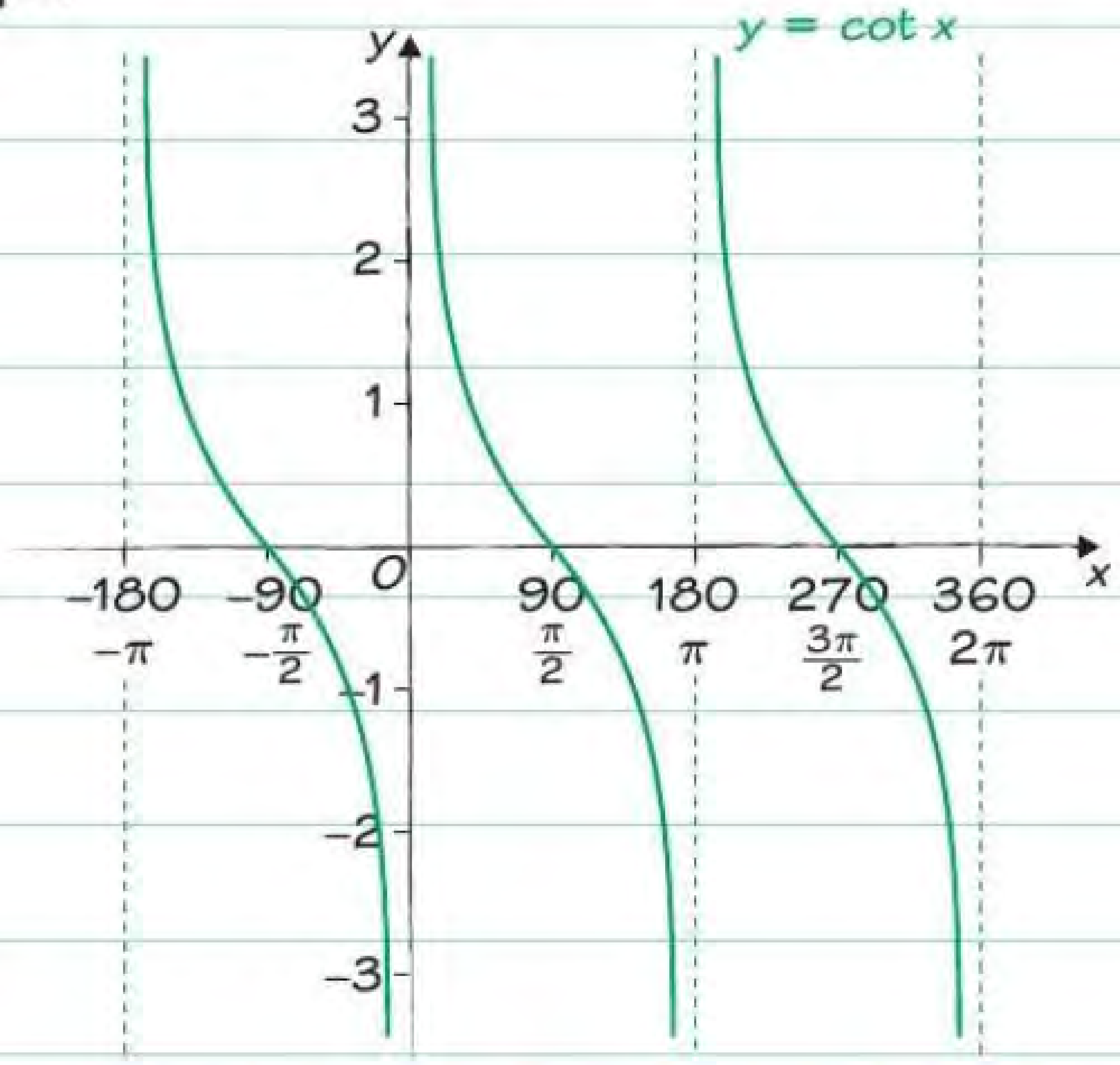
Use the **third letter** to remember the reciprocal trig functions:

coSec — $\dfrac{1}{\sin\theta}$ $\quad$ seC — $\dfrac{1}{\cos\theta}$ $\quad$ coT — $\dfrac{1}{\tan\theta}$

Graphs of sec, cosec and cot

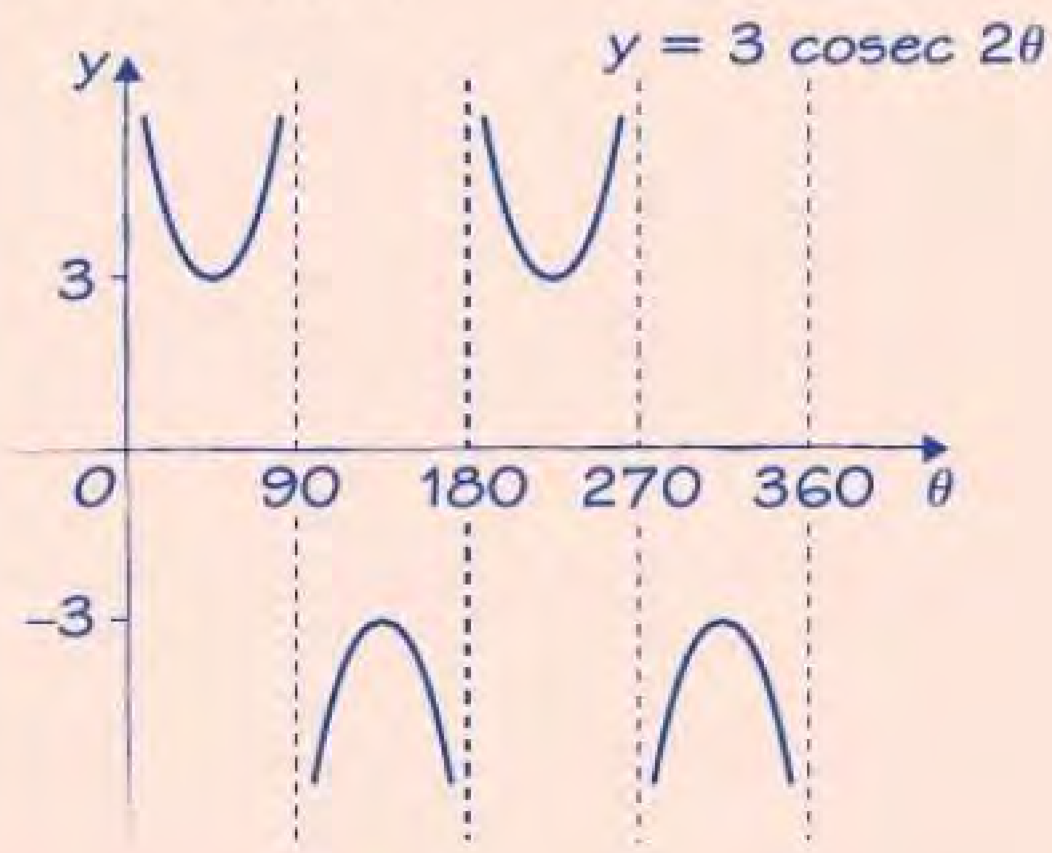
You need to be able to sketch these graphs in your exam. Don't rely on a graphing calculator – learn the **shapes**, **asymptotes** and **intercepts** for each graph.





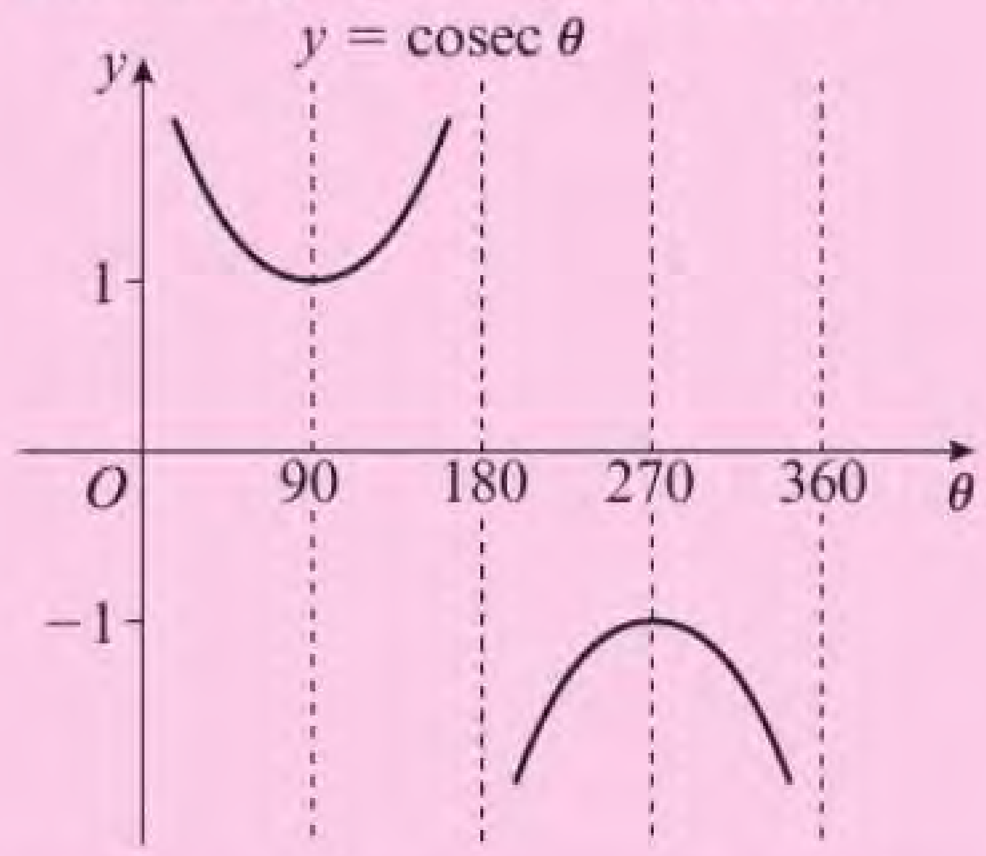
Worked example

Sketch the graph of $y = 3\operatorname{cosec}2\theta$ for $0° < \theta < 360°$. **(2 marks)**



Problem solved!

Start with the graph of $y = \operatorname{cosec}\theta$.



You need to apply two transformations:

- horizontal stretch, scale factor $\frac{1}{2}$
- vertical stretch, scale factor 3.

You will need to use problem-solving skills throughout your exam – **be prepared!**

Now try this

1 Sketch the graph of $y = \cot\left(x - \dfrac{\pi}{2}\right)$ for $0 \leqslant x \leqslant 2\pi$ **(2 marks)** Answer

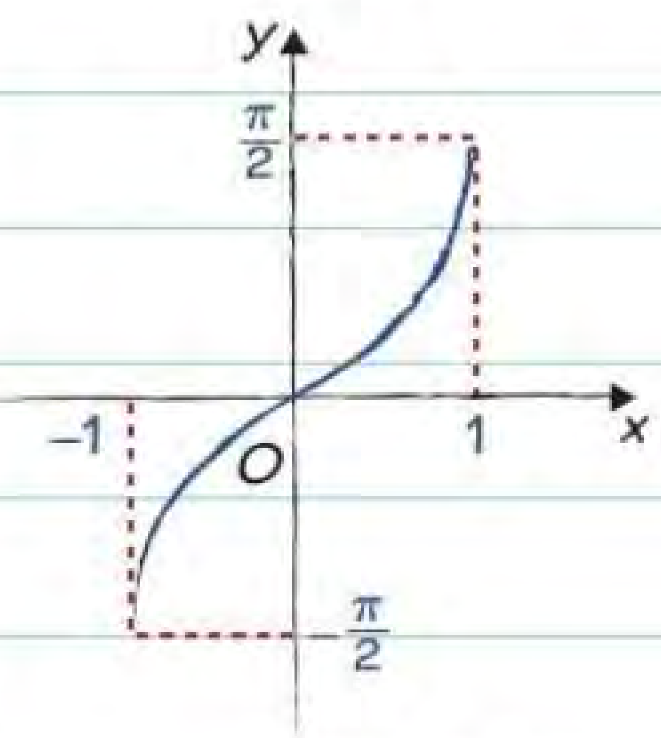
2 Sketch the graph of $y = \sec\frac{1}{2}\theta + 1$ for $-360° < \theta < 360°$ **(2 marks)** Answer

Look at page 13 for a reminder about transforming functions.

Arcsin, arccos and arctan

Arcsin, arccos and arctan are the mathematical names of the $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ functions on your calculator. They are the **inverse** functions of sin, cos and tan. You need to know their graphs, and their **domains** and **ranges**.

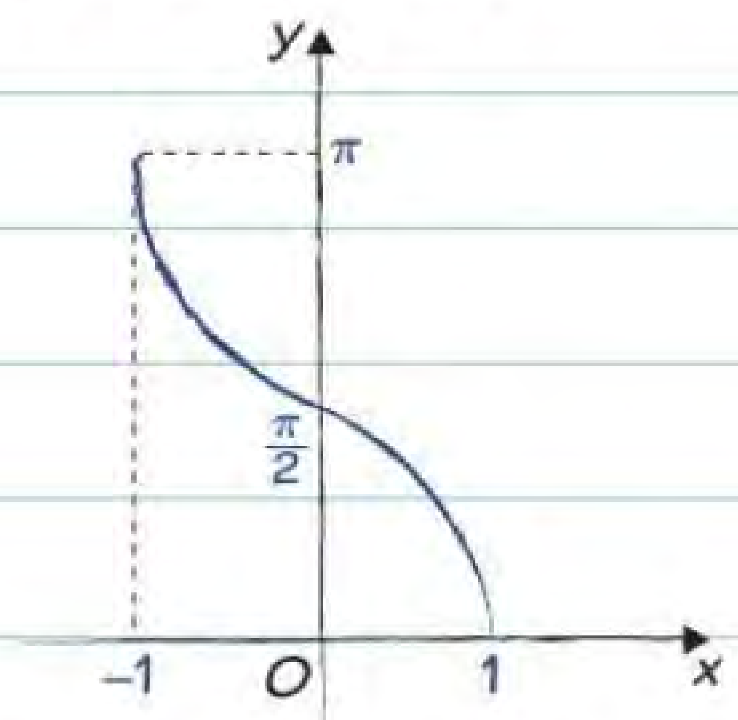
$y = \arcsin x$



Domain: $-1 \leqslant x \leqslant 1$

Range: $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$

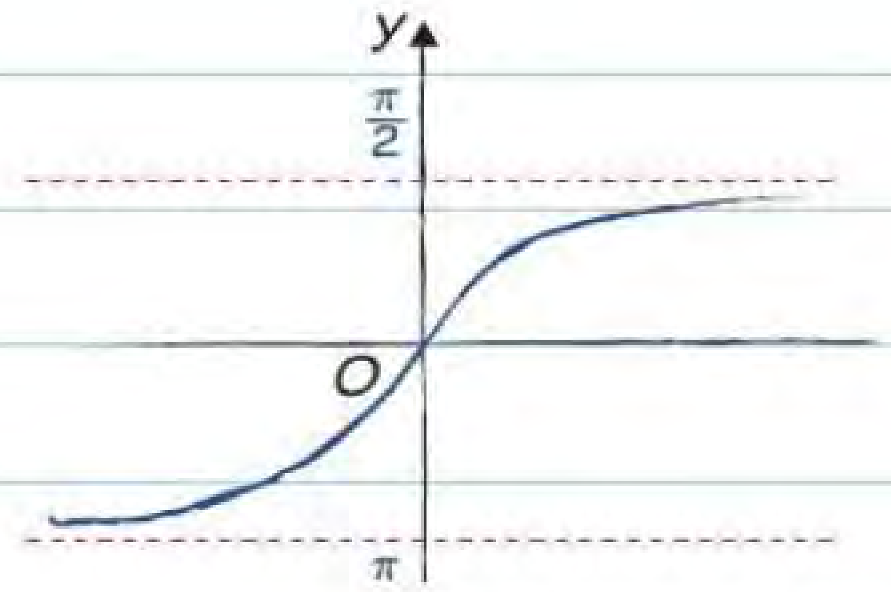
$y = \arccos x$



Domain: $-1 \leqslant x \leqslant 1$

Range: $0 \leqslant y \leqslant \pi$

$y = \arctan x$



Domain: $-\infty < x < \infty$ (or $x \in \mathbb{R}$)

Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Worked example

(a) Given that $y = \arcsin x$, express $\arccos x$ in terms of y. **(2 marks)**

$x = \sin y$

$x = \cos\left(\frac{\pi}{2} - y\right)$

$\arccos x = \frac{\pi}{2} - y$

(b) Hence evaluate $\arcsin x + \arccos x$. Give your answer in terms of π. **(2 marks)**

$\arcsin x + \arccos x = y + \frac{\pi}{2} - y$

$= \frac{\pi}{2}$

You should always use **radians** when you are working with arcsin, arccos and arctan in your exam.

Worked example

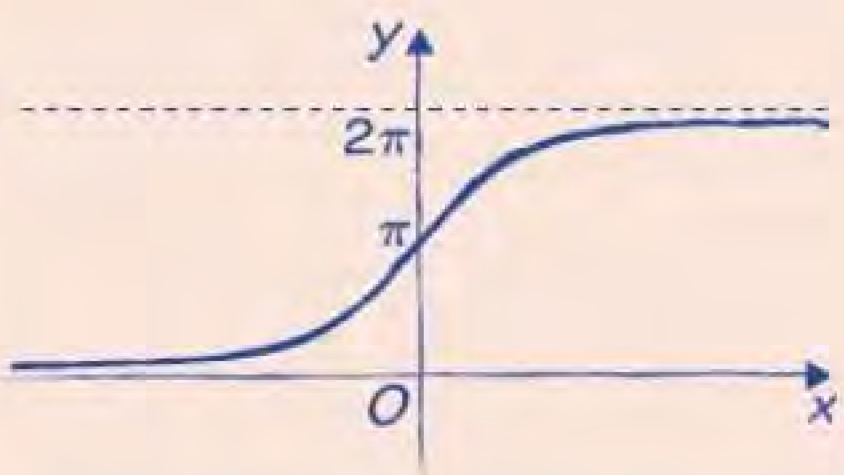
The function f is defined by

$f : x \mapsto \arctan x, \quad x \in \mathbb{R}$

(a) State the range of f. **(1 mark)**

$-\frac{\pi}{2} < f(x) < \frac{\pi}{2}$

(b) Sketch the graph of $y = 2f(x) + \pi$. Show clearly any asymptotes and points of intersection with the coordinate axes. **(2 marks)**



Now try this

1 Write down the value of the following in radians.

Answer (a) $\arctan\sqrt{3}$

Answer (b) $\arccos\frac{1}{2}$

Answer (c) $\arcsin\frac{1}{\sqrt{2}}$ **(3 marks)**

2 The function g is defined by

$g : x \mapsto \arcsin x - \frac{\pi}{4}, \quad x \in \mathbb{R}, \quad -1 \leqslant x \leqslant 1$

(a) Find $g\left(\frac{1}{2}\right)$, giving your answer in terms of π. Answer **(2 marks)**

(b) Solve the equation $g(x) = 0$, giving your answer as an exact value. **(3 marks)** Answer

(c) Find g^{-1} and state its domain. **(3 marks)** Answer

(d) Sketch the graph of $y = g^{-1}(x)$, showing the coordinates of the point where the graph crosses the x-axis. **(2 marks)** Answer

The domain of g^{-1} is the range of g. Start with the range of $\arcsin x$ then subtract $\frac{\pi}{4}$ from both limits.

Remember to sketch the graph of $y = g^{-1}(x)$ only for values of x in the domain of g^{-1}.