Summary of key points

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- The function $f(x)$ is **concave** on a given interval if and only if $f''(x) \leqslant 0$ for every value of x in that interval.
- The function $f(x)$ is **convex** on a given interval if and only if $f''(x) \geqslant 0$ for every value of x in that interval.

14 A **point of inflection** is a point at which $f''(x)$ changes sign.

Had a look ☐ Nearly there ☐ Nailed it! ☐

Differentiation and graphs

In your exam you might have to use $\frac{dy}{dx}$ to find the gradient of a curve at a given point. You can use this information to find turning points, and to find the equations of tangents and normals to the curve.

Worked example

A curve C has equation

$$y = \frac{3}{(5-3x)^2}, \quad x \neq \frac{5}{3}$$

The point P on C has x-coordinate 2. Find an equation of the normal to C at P in the form $ax + by + c = 0$, where a, b and c are integers. **(7 marks)**

$y = 3(5-3x)^{-2}$

$\frac{dy}{dx} = -6(5-3x)^{-3} \times (-3)$

$= 18(5-3x)^{-3}$

$= \frac{18}{(5-3x)^3}$

At P, $\frac{dy}{dx} = \frac{18}{(5-3(2))^3} = -18$

At P, $y = \frac{3}{(5-3(2))^2} = 3$

so P is the point $(2, 3)$

Gradient of normal at $P = \frac{-1}{-18} = \frac{1}{18}$

$y - 3 = \frac{1}{18}(x-2)$

$18y - 54 = x - 2$

$x - 18y + 52 = 0$

Tangents and normals

If the point $P(x_0, y_0)$ lies on the curve with equation $y = f(x)$:

1. The tangent at P has gradient $f'(x_0)$
   An equation for the tangent could be $y - y_0 = f'(x_0)(x - x_0)$
2. The normal at P has gradient $\frac{-1}{f'(x_0)}$
   An equation for the normal could be $y - y_0 = \frac{-1}{f'(x_0)}(x - x_0)$

Problem solved!

It's easier to use the **chain rule** than the **quotient rule** to differentiate a function like this. Rewrite it with a negative power.

Make sure you read the question carefully. You need to find the **normal** and not the **tangent**, and you will lose a mark if you don't give your final equation in the correct form.

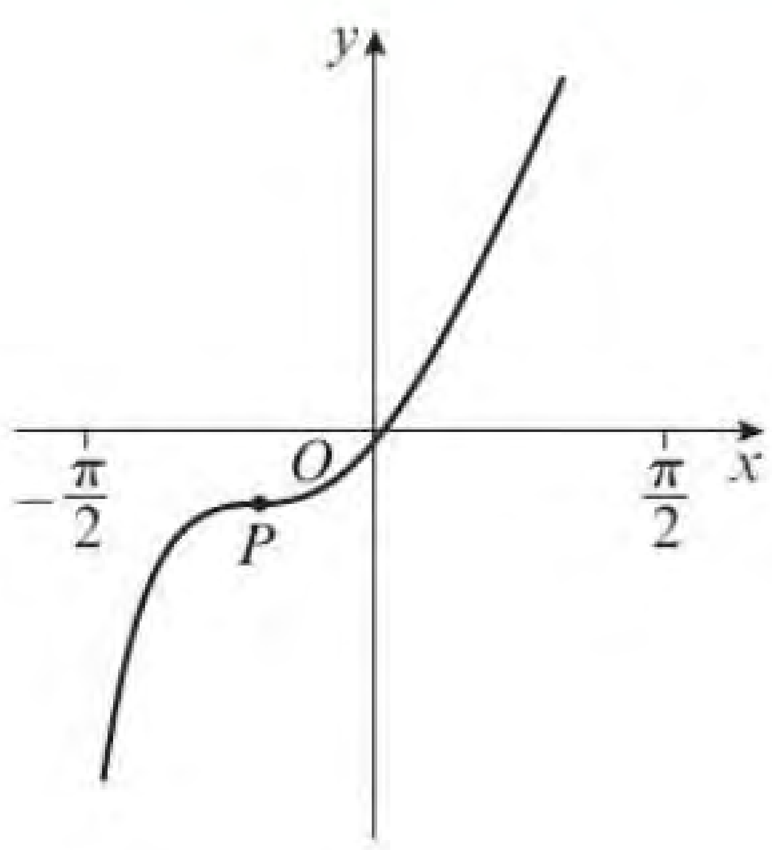
You will need to use problem-solving skills throughout your exam – **be prepared!**

The curve $y = f(x)$ has **turning points** when $\frac{dy}{dx} = 0$. To find the coordinates of P, you need to solve the equation $e^{2x}(1 + \tan x)^2 = 0$

Now try this

The diagram shows a sketch of the curve with equation $y = e^{2x}\tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

The curve has a turning point at P.



(a) Show that $\frac{dy}{dx} = e^{2x}(1 + \tan x)^2$ Answer **(3 marks)**

(b) Find the exact coordinates of P. Answer **(3 marks)**

(c) Find an equation of the tangent to the curve at the point where $x = 1$. Give your answer in the form $y = ax + b$ where a and b are constants given to 3 significant figures. Answer **(3 marks)**

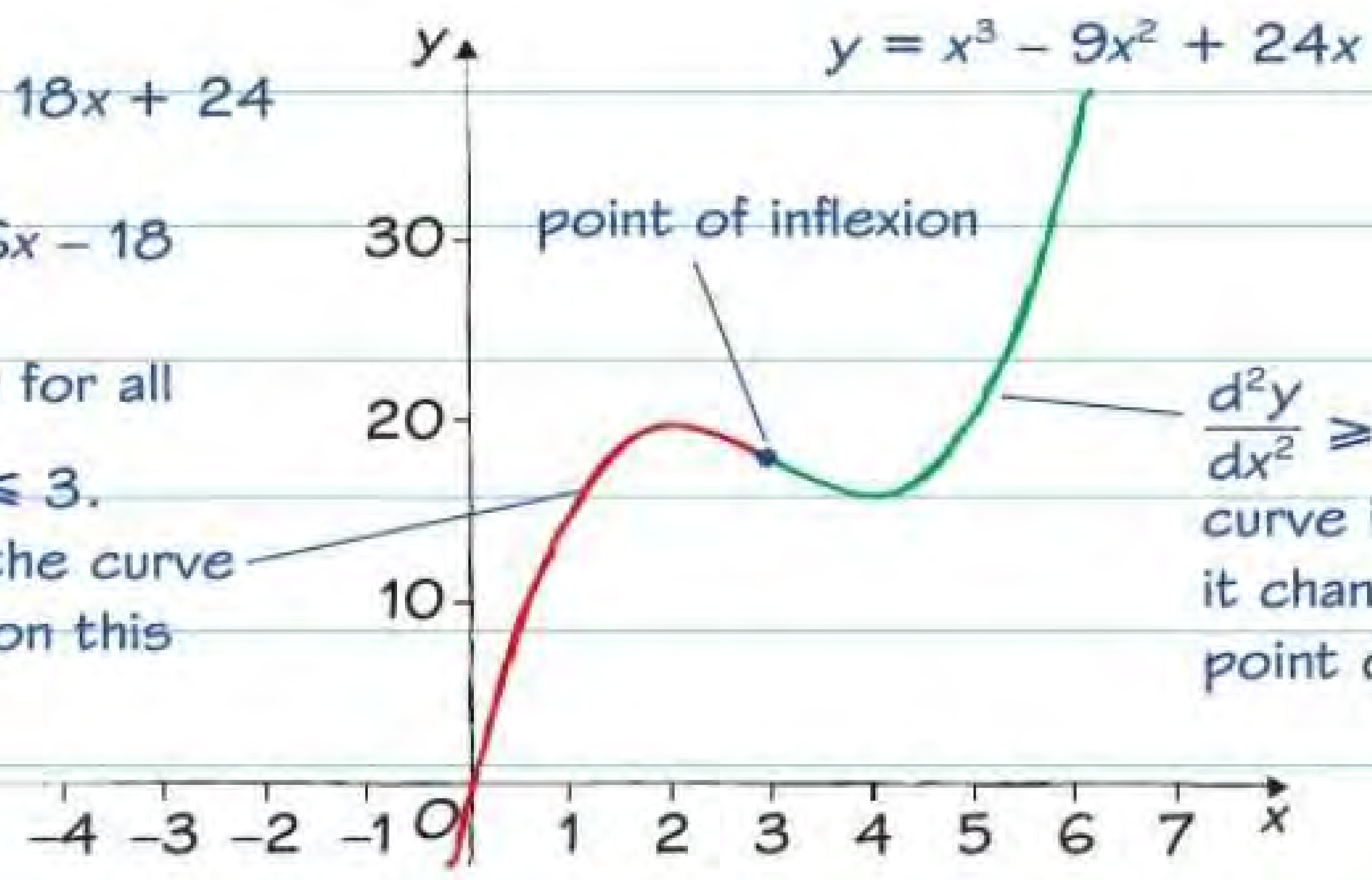
Points of inflexion

You can use **second derivatives** to determine where a curve is **concave** or **convex**, and to find points of inflexion. A point of inflexion occurs when the curve changes from being concave to convex, or vice versa.

$\frac{dy}{dx} = 3x^2 - 18x + 24$

and $\frac{d^2y}{dx^2} = 6x - 18$

So $\frac{d^2y}{dx^2} \leqslant 0$ for all values of $x \leqslant 3$. This means the curve is **concave** on this interval.



$\frac{d^2y}{dx^2} \geqslant 0$ for all values of $x \geqslant 3$. This means the curve is **convex** on this interval. The point where it changes from being concave to convex is the point of inflexion.

Golden rules

- The curve $y = f(x)$ is **concave** on the interval $[a, b]$ if and only if $f''(x) \leqslant 0$ for all $a \leqslant x \leqslant b$.
- The curve $y = f(x)$ is **convex** on the interval $[a, b]$ if and only if $f''(x) \geqslant 0$ for all $a \leqslant x \leqslant b$.

Worked example

The curve C has equation $y = xe^x$

(a) Show that C is convex on the interval $[0, 1]$. **(5 marks)**

$\frac{dy}{dx} = xe^x + e^x = (x + 1)e^x$

$\frac{d^2y}{dx^2} = (x + 1)e^x + e^x = (x + 2)e^x$

$e^x > 0$ for all $x \in \mathbb{R}$ and $x + 2 > 0$ for all $x \in [0, 1]$, so $\frac{d^2y}{dx^2} \geqslant 0$ for all $x \in [0, 1]$

so C is convex on that interval.

(b) Find the exact coordinates of the point of inflexion on C. **(3 marks)**

$\frac{d^2y}{dx^2} = 0 \Rightarrow x = -2$. When $x \leqslant -2$, $\frac{d^2y}{dx^2} \leqslant 0$ and when $x \geqslant -2$, $\frac{d^2y}{dx^2} \geqslant 0$ so $x = -2$ is a point of inflexion. At $x = -2$, $y = -2e^{-2}$, so the coordinates are $(-2, -2e^{-2})$.

Use the product rule twice.

Watch out!

There are two things to be careful of when finding points of inflexion:

- A stationary point on a graph can be a local maximum, a local minimum, or a point of inflexion. But, in general, a point of inflexion does not have to be a stationary point.
- Although $f''(x)$ must equal 0 at a point of inflexion, having $f''(x) = 0$ doesn't necessarily guarantee a point of inflexion. You also need the sign of $f''(x)$ to change on either side of that point.

For example, when $x = 0$, the curve $y = x^4$ has $\frac{d^2y}{dx^2} = 0$. But $x = 0$ is a local minimum on this curve, not a point of inflexion, because $\frac{d^2y}{dx^2}$ is never negative.

Now try this

1 The curve C has equation $y = x^3 + x^2 - x + 2$

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ Answer **(4 marks)**

(b) Find the coordinates of the stationary points on C and determine their nature. Answer **(5 marks)**

(c) Find the coordinates of the point of inflexion on C. Answer **(3 marks)**

2 The curve C has equation $y = (x^2 - 2)e^x$
Show that this curve has two stationary points and find their coordinates. Answer **(8 marks)**

3 The curve C has equation $y = x^4 + 3x^3 - 6x^2 + 2$

(a) Show that C has two points of inflexion, and find the values of x at these points. Answer **(7 marks)**

(b) State, with a reason, whether C is concave or convex on the interval $[1, 2]$. **(2 marks)** Answer