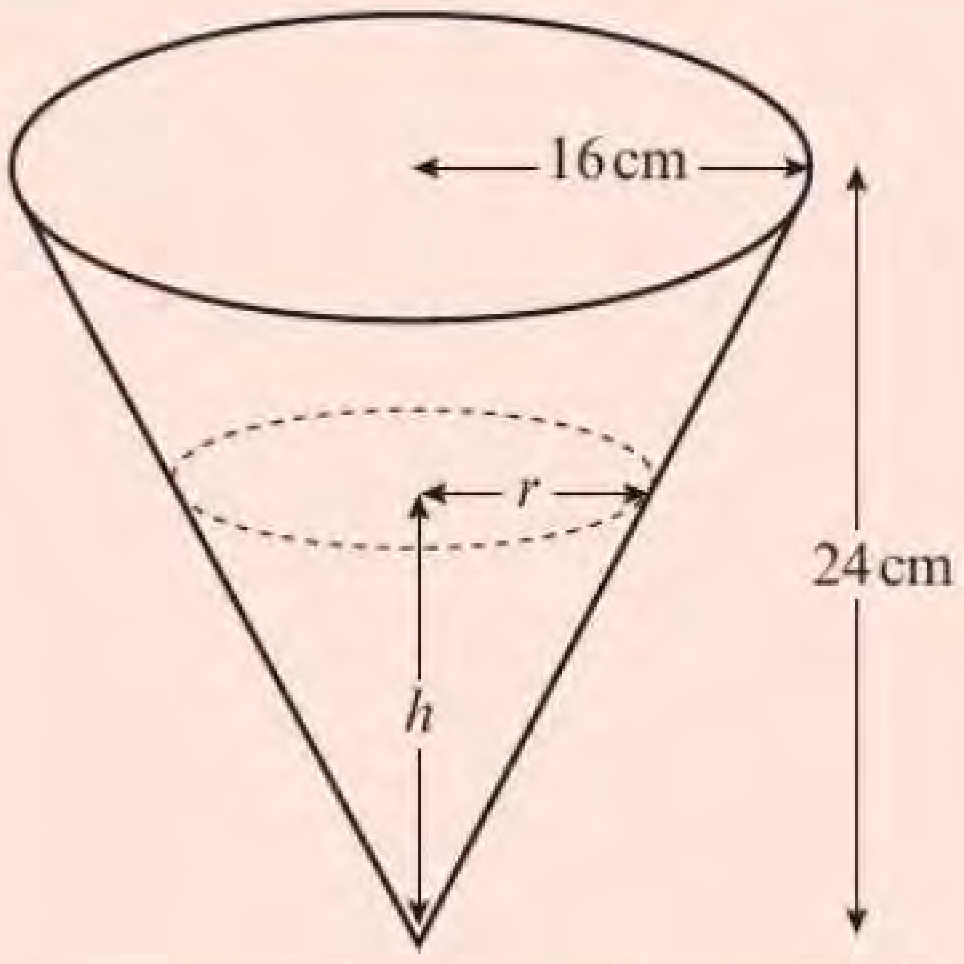
Rates of change

You can model lots of physical or financial situations by describing how a variable changes with time. Equations involving rates of change are called **differential equations**. You can revise how to **form** differential equations on this page, and how to **solve** them on page 109.

Worked example



A container is made in the shape of a hollow inverted right circular cone. The height of the container is 24 cm and the radius is 16 cm, as shown in the diagram. Water is flowing into the container. When the height of water is h cm, the surface of the water has radius r cm and the volume of water is $V\text{cm}^3$.

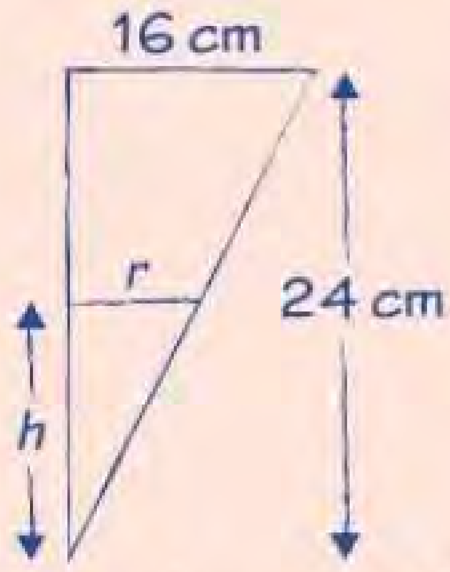
(a) Show that $V = \frac{4\pi h^3}{27}$ **(2 marks)**

[*The volume V of a right circular cone with vertical height h and base radius r is given by the formula* $V = \frac{1}{3}\pi r^2 h$.]

$\frac{r}{h} = \frac{16}{24}$ so $r = \frac{2h}{3}$

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{2h}{3}\right)^2 h$

$= \frac{4\pi h^3}{27}$



Water flows into the container at a rate of $8\text{ cm}^3\text{s}^{-1}$.

(b) Find, in terms of π, the rate of change of h when $h = 12$. **(5 marks)**

$\frac{dV}{dh} = \frac{12\pi h^2}{27} = \frac{4\pi h^2}{9}$

$\frac{dh}{dt} = \frac{dV}{dt} \div \frac{dV}{dh} = 8 \div \frac{4\pi h^2}{9}$

$= 8 \times \frac{9}{4\pi h^2}$

$= \frac{18}{\pi h^2}$

When $h = 12$:

$\frac{dh}{dt} = \frac{18}{\pi(12)^2} = \frac{1}{8\pi}$

If you need to use the formula for the volume of a cone in your exam it will be given to you with the question. You can use similar triangles to find the relationship between r and h. Draw a sketch to help you.

Chain rule

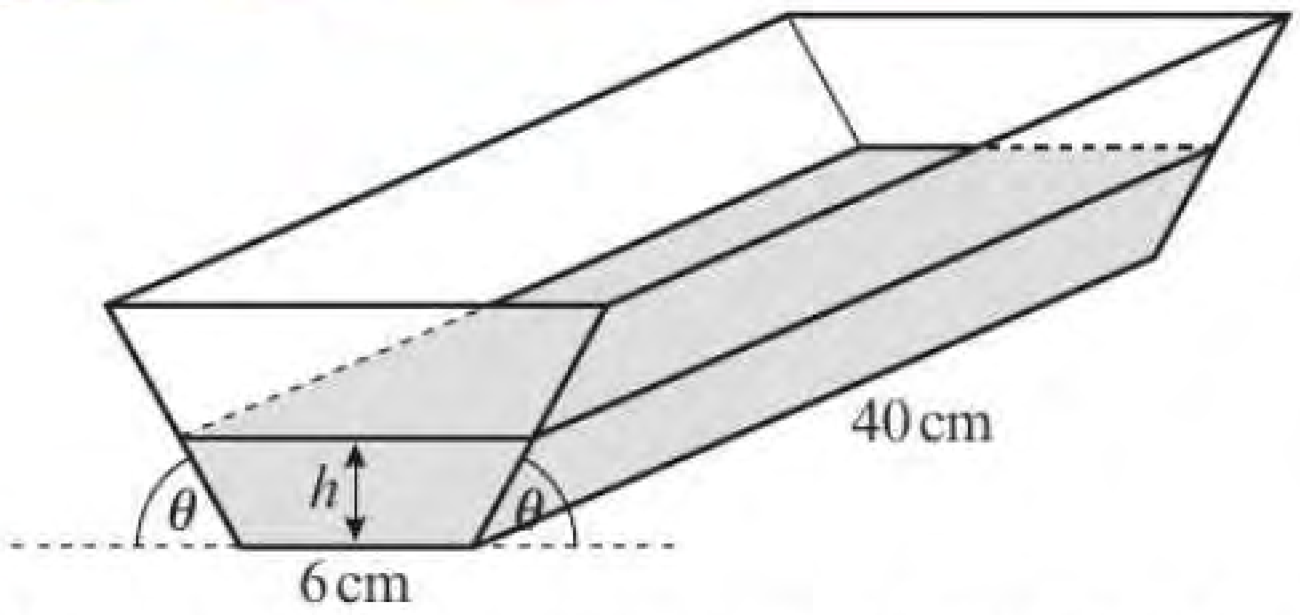
The chain rule allows you to multiply and divide derivatives in the same way as fractions.

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ so $\frac{dh}{dt} = \frac{dV}{dt} \div \frac{dV}{dh}$

Use your answer to part (a) to find $\frac{dV}{dh}$.
The rate the water is flowing into the container is $\frac{dV}{dt} = 8\text{ cm}^3\text{s}^{-1}$. The rate of change of h is $\frac{dh}{dt}$.

For part (a), remember that θ is a constant.

Now try this



The diagram shows a section of gutter in the shape of a prism. The cross-section of the gutter is a symmetrical trapezium. Water is flowing into the gutter. When the depth of the water is h cm, the volume of the water is $V\text{cm}^3$.

(a) Show that $\frac{dV}{dh} = 240 + 80h\cot\theta$ **(3 marks)**

Answer

Water flows into the gutter at a constant rate of $40\text{ cm}^3\text{s}^{-1}$.

(b) Given that when $h = 2.5$ the rate of change of h is 0.1 cm s^{-1}, find the value of θ correct to 1 decimal place. **(5 marks)**

Answer