Proof

Like an annoying child who keeps asking 'But whyyyyyyyy?', sometimes the examiners expect you to prove something is true. The next two pages feature two classic maths ways of proving things, plus a bonus way to disprove stuff.

Some Notation

There are certain bits of **notation** that'll be **useful** not only for **proofs**, but for the **rest** of A-Level Maths too.

A **set** is just a **collection** of objects or numbers (called **elements**), shown using **curly brackets**.
A set is often represented by a capital letter — e.g. A = {0, 1, 2}. There are different ways of writing sets:

- A list of elements — e.g. {1, 3, 5, 7, 9}
- A rule — e.g. {odd numbers between 0 and 10}
- Mathematical notation — e.g. $\{x : x < 0\}$ (this means "the set of values of x such that x is less than 0")

The symbols $\Rightarrow$ and $\Leftrightarrow$ are **logic symbols** — they show when one thing **implies** another.

- '$p \Rightarrow q$' means 'p implies q' or 'if p then q'. E.g. $x = 2 \Rightarrow x^2 = 4$.
- '$p \Leftrightarrow q$' means 'p implies q and q implies p' or 'p if and only if q'. E.g. $x^2 = 4 \Leftrightarrow x = \pm 2$

'If and only if' is sometimes written as 'iff'.

There are a few variations on the **equals sign** that you need to know:

- $\neq$ means **not equal to** — e.g. $\sin 90° \neq \cos 90°$
- $\approx$ means **approximately equal to** — e.g. $1 \div 3 \approx 0.33$
- $\equiv$ is the **identity symbol**. It means that two things are **identically equal** to each other.
  So $(a + b)(a - b) \equiv a^2 - b^2$ is true for all values of a and b
  (unlike an equation like $a^2 = 9$, which is only true for certain values of a).

Proof by Exhaustion

In **proof by exhaustion** you break things down into two or more **cases**. You have to make sure that your cases cover **all possible situations**, then prove **separately** that the statement is true for **each case**.

Example: Prove the following statement: "For any integer x, the value of $f(x) = x^3 + x + 1$ is an odd integer."

To prove the statement, split the situation into **two cases**:

(i) x is an **even number**, and (ii) x is an **odd number**

(i) If x is an **even integer**, then it can be written as $x = 2n$, for some integer n — *this is the definition of an even number.*

Substitute $x = 2n$ into the function: $f(2n) = (2n)^3 + 2n + 1 = 8n^3 + 2n + 1$
$= 2(4n^3 + n) + 1$

n is an integer $\Rightarrow (4n^3 + n)$ is an integer — *the sum or product of any integers are also integers.*
$\Rightarrow 2(4n^3 + n)$ is an even integer — *2 × an integer is the definition of an even number.*
$\Rightarrow 2(4n^3 + n) + 1$ is an **odd integer** — *since even + odd = odd.*

So $f(x)$ is **odd** when x is **even**.

(ii) If x is an **odd integer**, then it can be written as $x = 2m + 1$, for some integer m — *this is the definition of an odd number.*

Substitute $x = 2m + 1$ into the function:
$f(2m + 1) = (2m + 1)^3 + (2m + 1) + 1 = (8m^3 + 12m^2 + 6m + 1) + 2m + 1 + 1$
$= 8m^3 + 12m^2 + 8m + 3 = 2(4m^3 + 6m^2 + 4m + 1) + 1$

You can use the binomial expansion formula on p.51 to help you find these coefficients.

m is an integer $\Rightarrow (4m^3 + 6m^2 + 4m + 1)$ is an integer
$\Rightarrow 2(4m^3 + 6m^2 + 4m + 1)$ is an even integer
$\Rightarrow 2(4m^3 + 6m^2 + 4m + 1) + 1$ is an **odd integer**

So $f(x)$ is **odd** when x is **odd**.

We have shown that $f(x)$ is **odd** when x is even **and** when x is odd. As any integer x **must** be either odd or even, we have therefore shown that $f(x)$ is **odd** for **any** integer x.

Proof

Proof by **Deduction**

A **proof by deduction** (or **direct proof** or '**proof by direct argument**') is when you use **known facts** to **build up** your argument and show a statement **must** be true.

Example: A definition of a rational number is: 'a number that can be written as a quotient of two integers, where the denominator is non-zero'.
Use this definition to prove that the following statement is true:
"The product of two rational numbers is always a rational number."

Take **any two** rational numbers, call them a and b.

By the **definition** of rational numbers you can write them in the form $a = \frac{p}{q}$ and $b = \frac{r}{s}$, where p, q, r and s are all integers, and q and s are non-zero.

The **product** of a and b is $ab = \frac{p}{q} \times \frac{r}{s} = \frac{pr}{qs}$

pr and qs are the products of integers, so they must also be integers, and because q and s are non-zero, qs must also be non-zero.

We've shown that ab is a quotient of two integers and has a non-zero denominator, so by definition, **ab is rational**.

Hence the original statement is **true**.

Disproof by **Counter-Example**

Disproof by **counter-example** is the easiest way to show a mathematical statement is **false**. All you have to do is find **one case** where the statement doesn't hold.

Example: Disprove the following statement:
"For any pair of real numbers x and y, if $x > y$, then $x^2 + x > y^2 + y$."

To **disprove** the statement, it's enough to find just **one example** of x and y where $x > y$, but $x^2 + x \le y^2 + y$.

Let $x = 2$ and $y = -4$.

Then $2 > -4$, so $x > y$

but $x^2 + x = 2^2 + 2 = 6$ and $y^2 + y = (-4)^2 + (-4) = 12$, so $x^2 + x < y^2 + y$

So when $x = 2$ and $y = -4$, the first part of the statement holds, but the second part of the statement doesn't.

So the statement is **not true**.

Practice Questions

Q1 Write out the following sets as lists of elements:
a) {even prime numbers} b) {factors of 28} c) $\{x: x^2 = 1\}$

Q2 Disprove the following statements by giving a counter-example:
a) If $\frac{x}{y} < 1$, then $x < y$ for all values of x and y b) If $x^2 + y = y^2 + x$, then $x = y$

Exam Questions

Q1 Prove that, for any integer n, $(n + 6)^2 - (n + 1)^2$ is always divisible by 5. [3 marks]

Q2 Prove that the difference between an integer and its square is always even. [3 marks]

Q3 Is the following statement true or false? $(x + 2)(x - 1) > 2x - 2$ for all values of x.
Give either a proof or a counter-example. [3 marks]

If you've exhausted all options, just stick it in a proving drawer for an hour...

When you're trying to disprove something, don't be put off if you can't find a counter-example straight away. Sometimes you have to just try a few different cases until you find one that doesn't work.