Simultaneous Equations

Solving simultaneous equations is one of those things that you'll have to do again and again — so it's definitely worth practising them until you feel really confident.

Solving ***Simultaneous Equations*** *by* ***Elimination***

Solving **simultaneous equations** means finding the answers to two equations **at the same time** — i.e. finding values for x and y for which both equations are true.

Example: Solve the following equations: $3x + 5y = -4$
$-2x + 3y = 9$

Label the equations 1 and 2.

Go for the lowest common multiple (LCM). e.g. LCM of 2 and 3 is 6

1) Match the coefficients

Multiply the equations by numbers that will make either the x's or the y's match in the two equations (ignoring minus signs):

$$\text{①}\quad 3x + 5y = -4 \qquad \text{①} \times 2 \quad 6x + 10y = -8 \quad \text{③}$$
$$\text{②}\quad -2x + 3y = 9 \quad \Rightarrow \quad \text{②} \times 3 \quad -6x + 9y = 27 \quad \text{④}$$

Label these 3 and 4.

2) Eliminate to find one variable

The coefficients of x have different signs, so you need to add the equations (if the coefficients have the same sign, you'll need to subtract one equation from the other):

$$\text{③} + \text{④} \quad 6x + 10y + (-6x) + 9y = -8 + 27$$
$$\Rightarrow 19y = 19$$
$$\Rightarrow y = 1$$

3) Find the variable you eliminated

Put y = 1 into one of the equations to find x:

$$y = 1 \text{ in ①} \quad 3x + 5 = -4 \Rightarrow 3x = -9$$
$$\Rightarrow x = -3$$

4) Check your answer

Put x = –3 and y = 1 into the other equation:

$$x = -3, y = 1 \text{ in ②} \quad -2(-3) + 3(1) = 9 \ ✓$$

Use ***Substitution*** *if one equation is* ***Quadratic***

Sadly elimination won't always work. Sometimes one of the equations has not just x's and y's in it — but bits with x^2 and y^2 as well. When this happens, you can **only** use the **substitution** method:

One Quadratic and One Linear Equation

1) Isolate one variable in the linear equation by rearranging to get either x or y on its **own**.
2) **Substitute** the variable into the quadratic equation (to get an equation in just one variable).
3) Solve to get values for **one variable** — either by factorising or using the quadratic formula.
4) Stick these values in the linear equation to find the **corresponding values** for the other variables.

Substitute this for your own hilarious caption.

Example: Solve $-x + 2y = 5$ and $x^2 + y^2 = 25$.

Call the linear equation L and the quadratic equation Q.

Rearrange the linear equation so that either x or y is on its own on one side of the equals sign:

$$\text{Ⓛ}\quad -x + 2y = 5 \Rightarrow x = 2y - 5$$

Substitute this expression into the quadratic equation:

$$\text{Ⓠ}\quad x^2 + y^2 = 25 \Rightarrow (2y - 5)^2 + y^2 = 25$$

Rearrange this into the form $ax^2 + bx + c$ and then solve it:

$$\Rightarrow (4y^2 - 20y + 25) + y^2 = 25$$
$$\Rightarrow 5y^2 - 20y = 0$$
$$\Rightarrow 5y(y - 4) = 0$$
$$\Rightarrow y = 0 \text{ or } y = 4$$

$x^2 + y^2 = 25$ is actually a circle about the origin with radius 5 (see p.38).

Finally put both these values back into the linear equation to find corresponding values of x:

When $y = 0$: Ⓛ $-x + 2(0) = 5 \Rightarrow x = -5$
When $y = 4$: Ⓛ $-x + 2(4) = 5 \Rightarrow x = 3$
So the solutions are $x = -5, y = 0$ and $x = 3, y = 4$.

Check your answers by putting them back into Ⓛ and Ⓠ:

$x = -5, y = 0 \Rightarrow -(-5) + 2 \times 0 = 5$ and $(-5)^2 + 0^2 = 25$ ✓
$x = 3, y = 4 \Rightarrow -3 + 2 \times 4 = 5$ and $3^2 + 4^2 = 25$ ✓

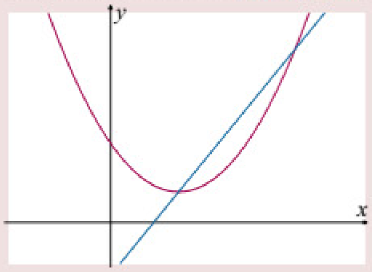
Simultaneous Equations

Number of **Solutions** *= number of* **Intersections**

When you have to interpret something **geometrically**, you have to sketch the graphs of the two functions and 'say what you see'. The number of **solutions** affects what your picture will look like:

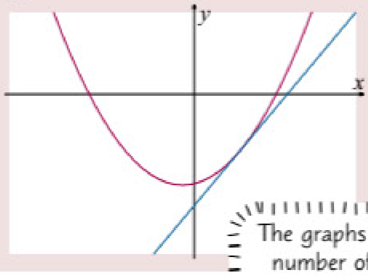
Two Solutions

The graphs meet in **two places**.



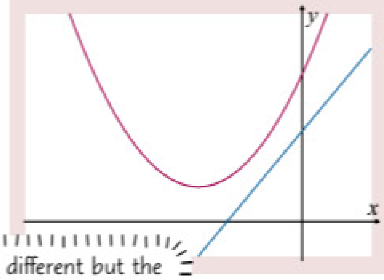
One Solution

The graphs meet in **one place** — the straight line is a **tangent** to the curve.



No Solutions

The graphs **never meet**.



The graphs can look different but the number of intersections will always equal the number of solutions.

Example: Interpret geometrically: $y = x^2 - 4x + 5$
$y = 2x - 4$

Substitute the expression for y from (2) into (1): $2x - 4 = x^2 - 4x + 5$

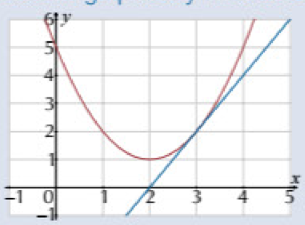
Rearrange and solve: $x^2 - 6x + 9 = 0$
$(x - 3)^2 = 0$
$\Rightarrow x = 3$

Putting x = 3 in (2) gives: $x = 3 \Rightarrow y = 2 \times 3 - 4 = 2$

There's only 1 solution: $x = 3, y = 2$

Geometric Interpretation

Since the equations have only one solution, the two graphs only meet at one point — (3, 2):



The straight line is a tangent to the curve at (3, 2).

Practice Questions

Q1 Solve these sets of simultaneous equations:

a) $3x - 4y = 7$ and $-2x + 7y = -22$ b) $2x - 3y = \frac{11}{12}$ and $x + y = -\frac{7}{12}$

Q2 Find where the following lines meet:

a) $y = 3x - 4$ and $y = 7x - 5$ b) $y = 13 - 2x$ and $7x - y - 23 = 0$ c) $2x - 3y + 4 = 0$ and $x - 2y + 1 = 0$

Q3 Find the possible solutions to these sets of simultaneous equations. Interpret your answers geometrically.

a) $y = x^2 - 7x + 4$
$2x - y - 10 = 0$

b) $y = 30 - 6x + 2x^2$
$y = 2(x + 11)$

c) $x^2 + 2y^2 - 3 = 0$
$y = 2x + 4$

Exam Questions

Q1 Find the coordinates of the points of intersection of $x^2 + 2y^2 = 36$ and $x + y = 6$. [6 marks]

Q2 The line l has equation $y = 2x - 3$ and the curve C has equation $y = (x + 2)(x - 4)$.

a) Sketch the line l and the curve C on the same axes, showing the coordinates of the x- and y- intercepts. [4 marks]

b) Show that the x-coordinates of the points of intersection of l and C satisfy the equation $x^2 - 4x - 5 = 0$. [2 marks]

c) Hence, or otherwise, find the points of intersection of l and C. [2 marks]

The Eliminator — a robot sent back through time to destroy one variable...

For a linear and quadratic equation you have to use substitution, but for a pair of linear equations elimination is usually the easiest method. Knowing how to sketch a graph is handy here, but there's no need to be a Van Gogh about it. Simultaneous equations pop up all over A-level maths so it's worth spending some time now to get them sorted.

Inequalities

Solving inequalities is very similar to solving equations. You've just got to be really careful that you keep the inequality sign pointing the right way.

*When **Multiplying** or **Dividing** by something **Negative** flip the inequality sign*

Like I said, these are pretty similar to solving equations — because whatever you do to one side, you have to do to the other. But multiplying or dividing by **negative** numbers **changes** the direction of the inequality sign.

Adding or **subtracting** doesn't change the direction of the inequality sign.

Multiplying or **dividing** by a **positive** number doesn't affect the inequality sign.

Examples: Find the range of values of x that satisfies: a) $x - 3 < -1 + 2x$ b) $8x + 2 \geq 2x + 17$

a) Adding 1 to both sides leaves the inequality sign pointing in the same direction.

$x - 3 < -1 + 2x$

$\Rightarrow x - 2 < 2x$

$\Rightarrow -2 < x$

Subtracting x from both sides doesn't affect the inequality.

so $x > -2$

b) Subtract 2, and then 2x, from both sides.

$8x + 2 \geq 2x + 17$

$\Rightarrow 8x \geq 2x + 15$

$\Rightarrow 6x \geq 15$

Divide both sides by 6 and simplify.

$\Rightarrow x \geq \frac{5}{2}$

Multiplying or **dividing** an inequality by a **negative** number changes the direction of the inequality sign.

Example: Find the range of values of x that satisfies $4 - 3x \leq 16$

Subtract 4 from both sides.

$4 - 3x \leq 16$

$\Rightarrow -3x \leq 12$

Then divide both sides by −3 — but change the direction of the inequality.

$\Rightarrow x \geq -4$

The reason for the sign changing direction is because it's just the same as swapping everything from one side to the other: $-3x \leq 12 \Rightarrow -12 \leq 3x \Rightarrow x \geq -4$

***Sketch** a **Graph** to solve a **Quadratic** inequality*

With quadratic inequalities, you're best off sketching the **graph** and taking it from there.
You've got to be really careful when you divide by variables that might be **negative** — basically, don't do it.

Example Find the range of values of x that satisfies $36x \leq 6x^2$

Rearrange into the form $f(x) \geq 0$. Start by dividing by 6: Take 6x from both sides and rearrange:

$36x \leq 6x^2$

$\Rightarrow 6x \leq x^2$

$\Rightarrow 0 \leq x^2 - 6x$

So $x^2 - 6x \geq 0$

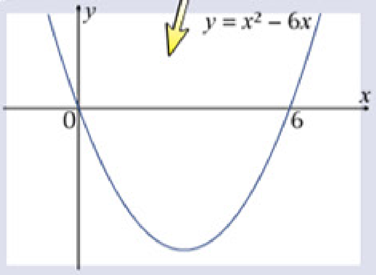
You shouldn't divide by x here because it could be negative (or zero).

Write the inequality as an equation: Let $y = x^2 - 6x$

Find where the curve crosses the x-axis by setting y = 0 and factorising: $x^2 - 6x = 0 \Rightarrow x(x - 6) = 0$
So $x = 0$ and $x = 6$.

Use a sketch to find where $x^2 - 6x \geq 0$: The graph is positive to the left of $x = 0$ and to the right of $x = 6$ (inclusive).
So $x \leq 0$ or $x \geq 6$.

The coefficient of x^2 is positive, so the graph of y is u-shaped.



You might be asked to give your answer in **set notation**:

In set notation, the answer to the example above is $\{x : x \leq 0\} \cup \{x : x \geq 6\}$.

Set Notation

- Set notation uses **curly brackets**: $\{x : x < a\}$ means 'the set of values of x such that x is less than a'.
- The **empty set**, written $\varnothing$, contains **nothing**. For example, $\{x : x^2 < 0\} = \varnothing$ (as x^2 is never < 0).
- The **union** ($\cup$) of two sets is **everything** contained in **either set**: $x < a$ **or** $x > b$ is written as $\{x : x < a\} \cup \{x : x > b\}$.
- The **intersection** ($\cap$) of two sets is **only** the things present in **both sets**: $x > c$ **and** $x < d$ is written as $\{x : x > c\} \cap \{x : x < d\}$.
- You can also use **brackets** to show an interval — **round** means the value **isn't** included, and **square** means it **is**. So $(3, 5]$ means $3 < x \leq 5$.

Inequalities

Test *Both Sides* of a curve to find a *Region*

You might be given two (or more) inequalities and asked to find the **region** that satisfies them. All you need to do is **sketch** the curves or lines and **test** the coordinates of a point (usually the origin) in each of the inequalities. The region you're after will include the **areas** where each inequality holds **true** for any tested point.

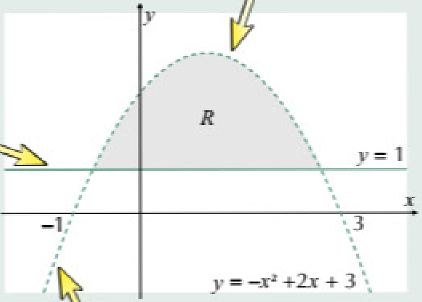
Example Draw and label the region that satisfies the following inequalities:
$y < -x^2 + 2x + 3$ and $y \geq 1$.

The coefficient of x^2 is negative so the curve is n-shaped.

Write them as equations: $y = -x^2 + 2x + 3$ and $y = 1$

Draw them on the same graph: $y = (x + 1)(3 - x) \Rightarrow y = 0$ at $x = -1$ and $x = 3$,
$y = 1$ is a horizontal straight line.

For inequalities with ≤ and ≥ you need to use a solid line.



Work out which side of each line you want by testing the origin in the inequalities:

$y < -x^2 + 2x + 3 \Rightarrow 0 < 3$, which is true.
Since the origin is below the curve, the region includes the area **below $y = -x^2 + 2x + 3$**.

$y \geq 1 \Rightarrow 0 \geq 1$, which is false.
Since the origin is below the line, the region also includes the area **above $y = 1$**.

If the curve goes through the origin you'd have to use a different point.

For inequalities with < and > you need to use a dotted line.

Label the correct region: The region, R, is the intersection of these areas.

Practice Questions

Q1 Find the ranges of x that satisfy these inequalities:
a) $x + 6 < 5x - 4$ b) $4x - 2 > x - 14$ c) $7 - x \leq 4 - 2x$

Q2 Solve: a) $7x - 4 > 2x - 42$ b) $12y - 3 \leq 4y + 4$ c) $9y - 4 \geq 17y + 2$

Q3 Find the ranges of x that satisfy the following inequalities:
a) $3x^2 - 5x - 2 \leq 0$ b) $6 - x - 2x^2 < 0$ c) $3x^2 + 7x + 4 \geq 2(x^2 + x - 1)$

Q4 Find the ranges of x that satisfy these jokers. Give your answers in set notation.
a) $x^2 + 3x - 1 \geq x + 2$ b) $2x^2 > x + 1$ c) $3x^2 - 12 < x^2 - 2x$

Q5 Draw and label the region that satisfies the following inequalities:
$y \leq 3$, $y > x - 1$ and $y > 4 - 2x$.

Exam Questions

Q1 Find the set of values for x that satisfy the inequalities below.
a) $3x + 2 \leq x + 6$ [1 mark]
b) $20 - x - x^2 > 0$ [2 marks]
c) both $3x + 2 \leq x + 6$ and $20 - x - x^2 > 0$ [1 mark]

Q2 Solve the inequalities:
a) $3 \leq 2p + 5 \leq 15$ [2 marks]
b) $q^2 - 9 > 0$ [2 marks]

Q3 Draw and label the region that satisfies the following: $y > 2x^2 - x - 3$ and $y \geq 1 - \frac{1}{2}x$ [4 marks]

Inequalities ≥ vectors > biology...

For inequality questions you could be given any linear and / or quadratic inequalities to sketch (or find intersection points). Don't forget to use a solid line for inequalities with ≤ or ≥ and a dotted line for < or > — this could cost you marks in the exam. There are plenty of inequality questions to get stuck into here, so get cracking.