Solving Quadratic Equations

You've probably been solving equations since before you could walk, so lots of this should be familiar. Here, you'll be finding x for lovely quadratic equations of the form $ax^2 + bx + c = 0$.

Solve quadratic equations by **Factorising**

Factorising is probably the quickest way to solve a quadratic equation — if it looks fairly **simple**, try to factorise it. The examples below use the methods described on pages 8-9.

Example: Solve $x^2 - 8 = 2x$ by factorising.

Put into $ax^2 + bx + c = 0$ form: $x^2 - 8 = 2x \Rightarrow x^2 - 2x - 8 = 0$

Solve the equation by factorising:
$$(x + 2)(x - 4) = 0$$
$$\Rightarrow x + 2 = 0 \text{ or } x - 4 = 0$$
$$\Rightarrow \mathbf{x = -2} \text{ or } \mathbf{x = 4}$$

Watch out for '**disguised quadratics**', where there's some **function of** x instead of x. To solve quadratics of the form $a(f(x))^2 + b(f(x)) + c$ use the **substitution** $y = f(x)$, solve to find values of y, then use these to find values of x.

Example: Solve $x^{\frac{2}{3}} + 3x^{\frac{1}{3}} - 40 = 0$.

$f(x) = x^{\frac{1}{3}}$, so use the substitution: $y = x^{\frac{1}{3}}$
$$y^2 + 3y - 40 = 0$$
Using $x^{\frac{2}{3}} = (x^{\frac{1}{3}})^2 = y^2$

Solve the quadratic in y by factorising:
$$(y + 8)(y - 5) = 0$$
$$\Rightarrow y + 8 = 0 \text{ or } y - 5 = 0$$
$$\Rightarrow y = -8 \text{ or } y = 5$$

Use the values of y to find the values of x:
$$y = -8 \Rightarrow -8 = x^{\frac{1}{3}} \Rightarrow x = (-8)^3 = \mathbf{-512}$$
$$y = 5 \Rightarrow 5 = x^{\frac{1}{3}} \Rightarrow x = 5^3 = \mathbf{125}$$

'Disguised quadratics' might involve trig functions (see Section 5) or exponentials and logs (see Section 6).

The quadratics were becoming more cunning with their disguises.

Completing the Square puts any old quadratic in a **Special Form**

Completing the square sounds really confusing. For starters, what does "Completing the Square" **mean**? **What** is the square? **Why** does it need completing? Well, there is **some** logic to it:

1) The **square** looks like this: $(x + \text{something})^2$

   It's basically the factorised equation (with two identical factors), but there's something missing...

2) ...so you need to '**complete**' it by adding a number to the square to make it equal to the original equation.

   $(x + \text{something})^2 + ?$

You start with something like this... ...sort the x-coefficients... ...and end up with something like this.

$2x^2 + 8x - 5$ → $2(x + 2)^2 + ?$ → $2(x + 2)^2 - 13$

The method below can be used to complete the square of a quadratic expression:

Completing the Square of $ax^2 + bx + c$

① Take a **factor of a** out of the x^2 and x terms: $a\left(x^2 + \frac{b}{a}x\right) + c$.

② Rewrite the bit in the bracket as **one bracket squared**.
The number in the brackets is always $\frac{b}{2a}$, so the bracket is $a\left(x + \frac{b}{2a}\right)^2$.

③ **Add d** to the bracket to complete the square and **find d** by setting the new and original expressions **equal** to each other:

$$a\left(x + \frac{b}{2a}\right)^2 + d = ax^2 + bx + c$$

Expanding the LHS gives: $\cancel{ax^2} + \cancel{bx} + \frac{b^2}{4a} + d = \cancel{ax^2} + \cancel{bx} + c$. Then cancel and rearrange to get d.

④ **Solving** this equation always gives $d = \left(c - \frac{b^2}{4a}\right)$, so:

$$a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right) = ax^2 + bx + c$$

Solving Quadratic Equations

Complete the square to find *Exact Solutions*

Completing the square probably isn't the easiest way to solve an equation but it is useful if you are asked to **sketch a graph** or find an **exact solution** — usually this means **surds** will be involved.

Example: a) Rewrite $2x^2 - 8x + 3$ by completing the square.

You can also use the quadratic formula to find exact solutions (see p.20).

Take out a factor of 2 out of the x^2 and x terms: $2(x^2 - 4x) + 3$

Rewrite the bracket as one bracket squared: $b = -8$, so $\frac{b}{2a} = \frac{-8}{2 \times 2} = -2$, so the bracket is $2(x-2)^2$

Add d to the bracket. Then find d by setting the new and original equation equal to each other:

$$2(x-2)^2 + d$$
$$2(x-2)^2 + d = 2x^2 - 8x + 3$$
$$2x^2 - 8x + 8 + d = 2x^2 - 8x + 3$$

$8 + d = 3 \Rightarrow d = -5$, so the completed square is $\mathbf{2(x-2)^2 - 5}$

Using $d = c - \frac{b^2}{4a}$ also gives $d = 3 - \frac{8^2}{4 \times 2} = 3 - 8 = -5$.

b) Hence find exact solutions to $2x^2 - 8x + 3 = 0$.

Set your answer from part a) equal to 0 and solve to find x:

$$2(x-2)^2 - 5 = 0$$
$$\Rightarrow (x-2)^2 = \frac{5}{2}$$

There's a positive and negative square root: $\Rightarrow x - 2 = \pm\sqrt{\frac{5}{2}} = \pm\frac{\sqrt{10}}{2}$

Rationalise the denominator $\sqrt{\frac{5}{2}} = \frac{\sqrt{5}}{\sqrt{2}} = \frac{\sqrt{5}\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{\sqrt{10}}{2}$

You're asked for the exact solutions so leave your answer in surd form: $\mathbf{x = 2 + \frac{\sqrt{10}}{2}}$ and $\mathbf{x = 2 - \frac{\sqrt{10}}{2}}$

Sometimes it's best just to use the *Formula*

Factorising or completing the square can be really messy for some equations — in which case your best bet is to use the **quadratic formula** (see page 20). But be careful, the question probably won't tell you which method is best.

For example, if you are asked to solve $6x^2 + 87x - 144 = 0$ things can get tricky.

This will actually factorise, but there are 2 possible bracket forms to try:
$(6x \quad)(x \quad)$ or $(3x \quad)(2x \quad)$ For each of these, there are 8 possible ways of making 144 to try.

And completing the square would be much slower than using the formula.

Practice Questions

Q1 Solve the following equations. While you're doing this, sing a jolly song to show how much you enjoy it.
a) $x^2 + x - 12 = 0$ b) $2 + x - x^2 = 0$ c) $4x^2 - 1 = 0$ d) $3x^2 - 15x - 14 = 4x$

Q2 Solve $3(x+2)^2 - 17(x+2) - 6 = 0$ by substitution.

Q3 Solve these quadratic equations by completing the square, leaving your answers in surd form where necessary.
a) $x^2 - 6x + 5 = 0$ b) $3x^2 - 7x + 3 = 0$ c) $2x^2 - 6x - 2 = 0$ d) $x^2 + 4x + 6 = 12$

Q4 Solve the equation $5x^2 + 4x - 36 = x^2 - 3x$.

Exam Questions

Q1 a) Write $3x^2 + 2x - 2$ in completed square form. [3 marks]
b) Hence, or otherwise, solve the equation $3x^2 + 2x - 2 = 0$. Give your answers to 2 decimal places. [1 mark]

Q2 Find the exact solutions of the equation $6x^2 = 1 - 3x$, by completing the square. [4 marks]

I'm popular with squares — they always tell me how I complete them...

Completing the square is useful when you're sketching graphs (see page 19). It's worth making sure that you're comfortable with all the methods discussed here, they'll all come in handy at some point. If you've got a fancy calculator it might even solve quadratic equations for you — but don't forget to write some working out in the exam.

Quadratic Functions and Graphs

If a question doesn't seem to make sense, or you can't see how to go about solving a problem, try drawing a graph. It sometimes helps if you can actually see what the problem is, rather than just reading about it.

*Quadratic graphs are **Always** u-shaped or n-shaped*

The **coefficient of x^2** tells you whether a quadratic curve (called a **parabola**) is u-shaped or n-shaped. When sketching a graph you might have to consider the following things:

Sketching Quadratic Graphs

① **Up or down:** Decide on the shape of the curve
— if the coefficient of x^2 is positive, then the graph is u-shaped.
— if the coefficient of x^2 is negative, then the graph is n-shaped.

② **Axes:** Find where the curve crosses the y-axis (set $x = 0$) and x-axis (set $y = 0$).

③ **Maximum or minimum:** Find the maximum or minimum point by using the fact that it's halfway between the roots or by completing the square (see next page).

④ **Sketch the graph:** Make sure that you label all the bits that you need to.

A u-shaped graph has a minimum and a n-shaped graph has a maximum.

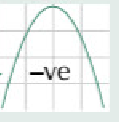
Examples:

Sketch $y = 8 - 2x - x^2$.

① **Up or Down**

$y = 8 - 2x - x^2$

The coefficient of x^2 is negative so the graph is n-shaped.



② **Axes**

When $x = 0$, $y = 8 - 2(0) - 0^2 = 8$

When $y = 0$, $8 - 2x - x^2 = 0$

$(2 - x)(x + 4) = 0$

$x = 2$ or $x = -4$

This means that the curve crosses the y-axis at (0, 8) and the x-axis at (2, 0) and (–4, 0).

③ **Max or Min**

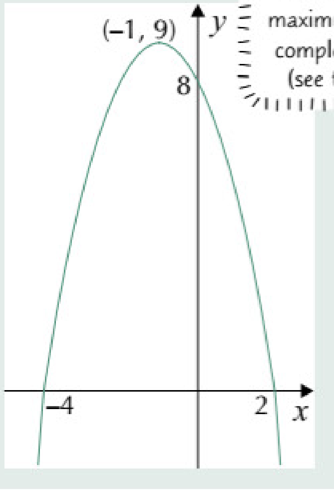
The maximum value is halfway between the roots because the curve is symmetrical:
$(2 + -4) \div 2 = -1$

So the maximum value is at $x = -1$.
The maximum is $y = 8 - 2(-1) - (-1)^2 = 9$

i.e. the graph has a maximum at the point $(-1, 9)$.

You can also find the maximum/minimum by completing the square (see the next page).

④ **Sketch**

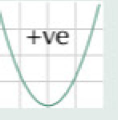


Sketch $y = 2x^2 - 5x + 3$.

① **Up or Down**

$y = 2x^2 - 5x + 3$

The coefficient of x^2 is positive so the graph is u-shaped.



② **Axes**

When $x = 0$, $y = 2(0^2) - 5(0) + 3 = 3$

When $y = 0$, $2x^2 - 5x + 3 = 0$

$(2x - 3)(x - 1) = 0$

$x = \frac{3}{2}$ or $x = 1$

This means that the curve crosses the y-axis at (0, 3) and the x-axis at $(\frac{3}{2}, 0)$ and (1, 0).

③ **Max or Min**

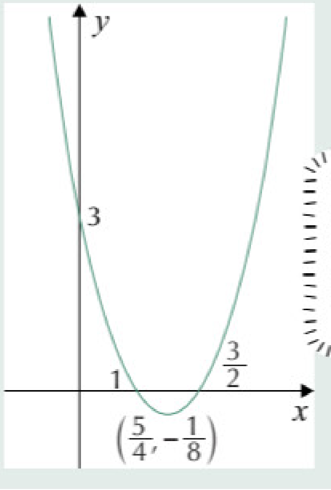
The minimum value is halfway between the roots:
$(\frac{3}{2} + 1) \div 2 = \frac{5}{4}$

The minimum value is at $x = \frac{5}{4}$.

The minimum is $y = 2\left(\frac{5}{4}\right)^2 - 5\left(\frac{5}{4}\right) + 3 = -\frac{1}{8}$

i.e. the graph has a minimum at the point $\left(\frac{5}{4}, -\frac{1}{8}\right)$.

④ **Sketch**



Sketches don't have to be drawn to scale but you need to get the shape right and label everything the question asks for.

Quadratic Functions and Graphs

Completing the square can be ***Useful***

Once you've completed the square, you can very quickly say **loads** about a quadratic function.
And it all relies on the fact that a squared number can **never** be less than zero... **ever**.

Example: Sketch the curve of $f(x) = 3x^2 - 6x - 7$.

Complete the square of f(x) (see page 16): $f(x) = 3x^2 - 6x - 7 = 3(x - 1)^2 - 10$

Find where y = f(x) crosses the axes:

When $x = 0$, $y = -7$, the curve crosses the y-axis at **−7**.

When $y = 0$, $3(x - 1)^2 - 10 = 0$

$$(x-1)^2 = \frac{10}{3}$$

$$x - 1 = \pm\sqrt{\frac{10}{3}}$$

$$x = 1 \pm \frac{\sqrt{30}}{3}$$

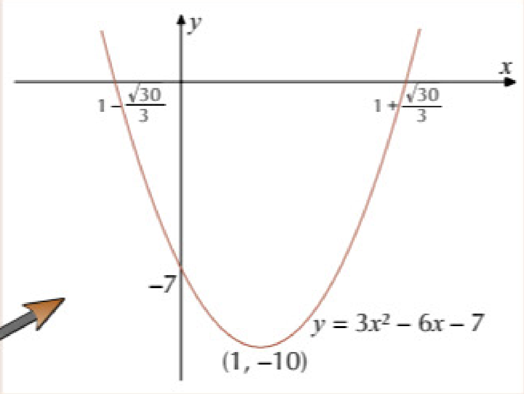
Rationalise the denominator (p.7).

So the curve crosses the x-axis at $x = 1 + \frac{\sqrt{30}}{3}$ and $x = 1 - \frac{\sqrt{30}}{3}$

$(x - 1)^2 \geq 0$ so the smallest value occurs when $(x - 1) = 0$, i.e. when $x = 1$.
Find the minimum by substituting $x = 1$ into f(x):

$f(x) = 3(x - 1)^2 - 10, \Rightarrow f(1) = 3(1 - 1)^2 - 10 = -10$
So the minimum is at **(1, −10)**.

Now you can sketch the graph.



Some functions don't have ***Real Roots***

By completing the square, you can also quickly tell if the graph of a quadratic function ever crosses the x-axis.
It'll only cross the x-axis if the function changes sign (i.e. goes from positive to negative or vice versa).

Example: $f(x) = x^2 + 4x + 7$. Does $f(x) = 0$ have any real roots?

The smallest this bit can be is zero (at $x = -2$). $f(x) = (x + 2)^2 + 3$

$(x + 2)^2$ is never less than zero, so the minimum of $f(x)$ is 3.

This means that:
a) $f(x)$ can never be negative.
b) The graph of $f(x)$ never crosses the x-axis.

So the function has **no real roots**.

To sketch a graph like this you'd need to show the correct shape, y-intercept, and coordinates of the minimum or maximum.

If the coefficient of x^2 is negative, you can do the same sort of thing to check whether f(x) ever becomes positive.

Practice Questions

Q1 Sketch the following curves, labelling the maximum/minimum and the points where it crosses the axes.
a) $y = x^2 + 4x - 5$ b) $y = 2x^2 - 7x + 6$ c) $y = 11x - 10 - 3x^2$ d) $y = 25 - 4x^2$

Q2 Show that the equation $f(x) = 0$, where $f(x) = 2x^2 - 12x + 23$, has no real roots.

Exam Questions

Q1 a) $f(x) = x^2 - 14x + k$, where k is a constant. Given that one of the roots of $f(x) = 0$ is $x = 7 + 2\sqrt{6}$, find the value of k, and hence verify that the other root is $x = 7 - 2\sqrt{6}$. [4 marks]
b) Sketch the curve of $y = f(x)$. Label the minimum and the points of intersection with the axes. [3 marks]

Q2 a) Rewrite $x^2 - 12x + 15$ in the form $(x - a)^2 + b$, for integers a and b. [2 marks]
b) Find the minimum value of $x^2 - 12x + 15$ and state the value of x at which this minimum occurs. [2 marks]

Sketches really help you get to the root of the problem...

Completing the square is really useful here — you can use it to find where the curve crosses the x-axis and also to find the minimum. Sketching graphs is probably the most fun you'll have in this section so draw away to your hearts content. And once you've had your fill of quadratics there's a load of other graphs to sketch later in the section.

The Quadratic Formula

Unlike factorising, the quadratic formula always works... no ifs, no buts, no butts, no nothing...

*I shall teach you the ways of the **Formula***

If you want to solve a quadratic equation $ax^2 + bx + c = 0$, then the answers are given by the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example: Solve the quadratic equation $3x^2 - 4x = 8$, leaving your answer in surd form.

Get the equation into the standard $ax^2 + bx + c = 0$ form: $3x^2 - 4x - 8 = 0$

Plug the values $a = 3$, $b = -4$, $c = -8$ into the formula (be very careful with all the minus signs):

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 3 \times -8}}{2 \times 3}$$

$$= \frac{4 \pm \sqrt{112}}{6} = \frac{2 \pm 2\sqrt{7}}{3}$$

Use the rules of surds (see p.6).

There are two answers (one using + and one using −): $x = \frac{2 + 2\sqrt{7}}{3}$ or $x = \frac{2 - 2\sqrt{7}}{3}$

Some **calculators** have a quadratic equation solver — you just enter the values of a, b and c and hey presto. This can be quite useful for **checking** your answers but make sure you show your working in the exam.

How Many Roots? *Check the $b^2 - 4ac$ bit...*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

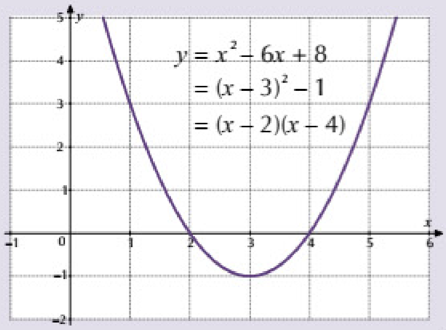
When you try to find the roots of a quadratic equation, this bit in the square root sign **($b^2 - 4ac$)** can be positive, zero, or negative. It's this that tells you if a quadratic equation has **two real roots**, **one real root**, or **no real roots**.

The $b^2 - 4ac$ bit is called the **discriminant** (sometimes written D, or Δ).

It's good to be able to picture what the graphs will look like in these different cases:

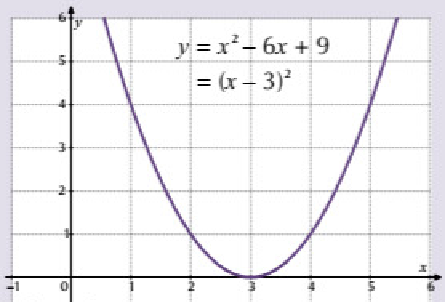
$b^2 - 4ac > 0$ — Two real roots

The graph crosses the x-axis twice and these values are the roots:



$b^2 - 4ac = 0$ — One real root

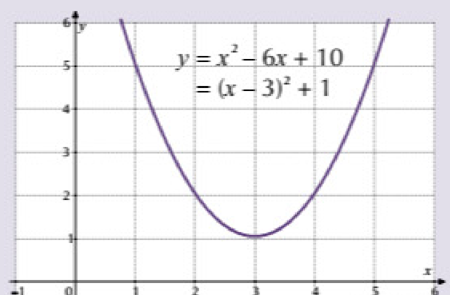
The graph just touches the x-axis from above (or from below if the x^2 coefficient is negative).



You might also see the term 'equal roots' — this means the same as 'one real root'.

$b^2 - 4ac < 0$ — No real roots

The graph doesn't touch the x-axis at all.



In some areas of maths, you can take the square root of negative numbers and get 'imaginary' numbers — that's why we say no 'real' roots when the discriminant is negative.

*Identify **a**, **b** and **c** to find the **Discriminant***

Make sure you get them the **right way round** — it's easy to get mixed up if the quadratic's in a **different order**.

Example: Find the discriminant of $15 - x - 2x^2$. How many real roots does $15 - x - 2x^2 = 0$ have?

First identify a, b and c: $a = -2$, $b = -1$ and $c = 15$ (NOT $a = 15$, $b = -1$ and $c = -2$)

Work out the discriminant: $b^2 - 4ac = (-1)^2 - (4 \times -2 \times 15) = 1 + 120 =$ **121**.

The discriminant is > 0: so $15 - x - 2x^2 = 0$ has **two distinct real roots**.

The Quadratic Formula

a, *b* and *c* might be *Unknown*

In exam questions, you might be given a **quadratic** where one or more of a, b and c are given in terms of an **unknown** (such as k, p or q). This means that you'll end up with an **equation** or **inequality** for the discriminant **in terms of the unknown —** you might have to **solve** it to find the **value** or **range of values** of the unknown.

Example: If $f(x) = 3x^2 + 2x + k$, find the range of values of k for which:
a) $f(x) = 0$ has 2 distinct roots, b) $f(x) = 0$ has 1 root, c) $f(x) = 0$ has no real roots.

Using a = 3, b = 2 and c = k, work out what the discriminant is:

$b^2 - 4ac = 2^2 - 4 \times 3 \times k = 4 - 12k$

The only difference is the (in)equality symbol.

a) Two distinct roots means:

$b^2 - 4ac > 0 \Rightarrow 4 - 12k > 0$
$\Rightarrow 4 > 12k$
$\Rightarrow k < \frac{1}{3}$

b) One root means:

$b^2 - 4ac = 0 \Rightarrow 4 - 12k = 0$
$\Rightarrow 4 = 12k$
$\Rightarrow k = \frac{1}{3}$

c) No real roots means:

$b^2 - 4ac < 0 \Rightarrow 4 - 12k < 0$
$\Rightarrow 4 < 12k$
$\Rightarrow k > \frac{1}{3}$

You might have to *Solve* a *Quadratic Inequality* to find k

When you put your values of a, b and c into the formula for the **discriminant**, you might end up with a **quadratic inequality** in terms of k. You'll have to solve this to find the range of values of k — there's more on this on p.24.

Example: The equation $kx^2 + (k + 3)x + 4 = 0$ has two distinct real solutions.
Show that $k^2 - 10k + 9 > 0$, and find the set of values of k which satisfy this inequality.

Using a = k, b = (k + 3) and c = 4, work out what the discriminant is:

$b^2 - 4ac = (k + 3)^2 - (4 \times k \times 4) = k^2 + 6k + 9 - 16k = k^2 - 10k + 9$

The equation has two distinct real solutions, so the discriminant must be > 0:

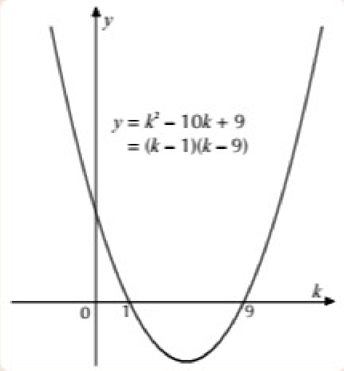
$k^2 - 10k + 9 > 0$

Now, to find the set of values for k, you have to factorise the quadratic:

$k^2 - 10k + 9 = (k - 1)(k - 9)$

This expression is zero when k = 1 and k = 9.
From the graph, you can see that this is a u-shaped quadratic which is > 0 when:

$k < 1$ or when $k > 9$



Practice Questions

Q1 For each of the following: (i) Find the discriminant and state the number of real roots of the quadratic.
(ii) Find the exact values of its real roots, if it has any.

a) $4x^2 + 28x + 49 = 0$ b) $3x^2 + 3x + 1 = 0$ c) $9x^2 - 6\sqrt{2}x + 2 = 0$ d) $2x^2 + 9x - 5 = 0$

Q2 If the quadratic equation $x^2 + kx + 4 = 0$ has two distinct real roots, find the possible values of k.

Exam Questions

Q1 The equation $x^2 + 2kx + 4k = 0$, where k is a non-zero integer, has equal roots. Find the value of k. [3 marks]

Q2 The equation $(p + 1)x^2 + (p + 1)x + 1 = 0$ has 2 distinct real solutions for x (p is a constant).

a) Show that $p^2 - 2p - 3 > 0$ [3 marks]

b) Hence find the range of possible values for p. [3 marks]

All the best mathematicians are raised on quadratic formula...

Don't panic if you're not sure how to solve quadratic inequalities — they're covered in more detail on page 24. Although it might be tempting to hide under your exam desk and hope a discriminant question doesn't find you, there's no escaping these questions — so get practising until you can recite the quadratic formula in your sleep.