Polynomials

A polynomial is just an expression of algebraic terms. In A-level maths you need to manipulate them all the time.

***Expand** brackets by **Multiplying** them out*

Here are the basic types you have to deal with — you'll have seen them all before.

Single Brackets

$a(b + c + d) = ab + ac + ad$

Double Brackets

$(a + b)(c + d) = ac + ad + bc + bd$

Squared Brackets

$(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2$

Use the middle stage until you're comfortable with it.
Just **never** make this **mistake**: $(a + b)^2 = a^2 + b^2$

Difference of Two Squares

$(a + b)(a - b) = a^2 - ab + ab - b^2 = a^2 - b^2$

The difference of two squares can be applied to surds: $(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y}) = x - y$

Long Brackets

Write it out again with **each term** from one bracket separately multiplied by the **other bracket**.

$(x + y + z)(a + b + c + d)$
$= x(a + b + c + d) + y(a + b + c + d) + z(a + b + c + d)$

Then **multiply out each** of these **brackets**, one at a time.

Example: Expand and simplify $(2x^2 + 3x + 6)(4x^3 + 6x^2 + 3)$

Multiply each term in the first bracket by the second bracket: $2x^2(4x^3 + 6x^2 + 3) + 3x(4x^3 + 6x^2 + 3) + 6(4x^3 + 6x^2 + 3)$

Multiply out each bracket individually: $= (8x^5 + 12x^4 + 6x^2) + (12x^4 + 18x^3 + 9x) + (24x^3 + 36x^2 + 18)$

Simplify it all: $= \mathbf{8x^5 + 24x^4 + 42x^3 + 42x^2 + 9x + 18}$

*Look for **Common Factors** when **Simplifying Expressions***

Something that is in each term of an expression is a **common factor** — this can be **numbers**, **variables** or even **brackets**. If you spot a common factor you can '**take it outside**' a bracket.

Example: Simplify $(x + 1)(x - 2) + (x + 1)^2 - x(x + 1)$

There's an $(x + 1)$ factor in each term, so we can take this out as a common factor (hurrah):

$$(x + 1)\{(x - 2) + (x + 1) - x\}$$

The terms inside the curly bracket are the old terms with an $(x + 1)$ removed.

At this point you should check that this multiplies out to give the original expression. (You can just do this in your head, if you trust it.)

Then simplify the big bracket's innards:

$$(x + 1)\{x - 2 + x + 1 - x\}$$
$$= (x + 1)(x - 1) = x^2 - 1$$

Use the "difference of two squares" (or multiply out) to get this answer

Factorise** a **Quadratic** by putting it into **Two Brackets

Factorising a quadratic in the form $ax^2 + bx + c$ is pretty easy when $a = 1$:

Factorising Quadratics

1) Write down the two brackets: $(x \quad)(x \quad)$
2) Find two numbers that **multiply** to give 'c' and **add / subtract** to give 'b' (ignoring signs).
3) Put the numbers in the brackets and choose the correct **signs**.

Example: Factorise $x^2 + 4x - 21$

1) $x^2 + 4x - 21 = (x \quad)(x \quad)$
2) 1 and 21 multiply to give 21 — and add / subtract to give 22 and 20.
   3 and 7 multiply to give 21 — and add / subtract to give 10 and **4**.
3) $x^2 + 4x - 21 = (x \quad 7)(x \quad 3)$
   $= \mathbf{(x + 7)(x - 3)}$

This is the value of 'b' you're after — 3 and 7 are the right numbers.

These get much easier with practice — you might even be able to do them in your head. Make sure you always check your answer by multiplying the brackets out.

Polynomials

Use a *Similar Method* for *Factorising* a quadratic when *a ≠ 1*

Example: Factorise $3x^2 + 4x - 15$

As before, write down two brackets — but instead of having x in each, you need two things that will multiply to give $3x^2$: $3x^2 + 4x - 15 = (3x \quad)(x \quad)$

It's got to be 3x and x here.

This is where it gets a bit fiddly. You need to find two numbers that multiply together to make 15 — but which will give you 4x when you multiply them by x and 3x, and then add/subtract them:

$(3x \quad 1)(x \quad 15) \Rightarrow x$ and $45x$ — which then add or subtract to give $46x$ and $44x$.

$(3x \quad 15)(x \quad 1) \Rightarrow 15x$ and $3x$ — which then add or subtract to give $18x$ and $12x$.

$(3x \quad 3)(x \quad 5) \Rightarrow 3x$ and $15x$ — which then add or subtract to give $18x$ and $12x$.

$(3x \quad 5)(x \quad 3) \Rightarrow 5x$ and $9x$ — which then add or subtract to give $14x$ and $4x$.

This is the value you're after — so this is the right combination.

You know the brackets must be like these... $(3x \quad 5)(x \quad 3) = 3x^2 + 4x - 15$ so all you have to do is put in the plus or minus signs:

'c' is negative — that means the signs in the brackets are different.

$(3x + 5)(x - 3) = 3x^2 - 4x - 15$

or...

$(3x - 5)(x + 3) = 3x^2 + 4x - 15$ ⟸ So it's this one.

You've only got two choices — if you're unsure, just multiply them out to see which one's right.

Simplify algebraic fractions by *Factorising* and *Cancelling Factors*

Algebraic fractions are a lot like normal fractions — and you can treat them in the **same way**, whether you're multiplying, dividing, adding or subtracting them. All fractions are much **easier** to deal with when they're in their **simplest form**, so the first thing to do with algebraic fractions is to **simplify** them as much as possible.

1) Look for **common factors** in the numerator and denominator — **factorise** top and bottom and see if there's anything you can **cancel**.

Examples: Simplify the following: a) $\frac{ax + ay}{az}$ b) $\frac{3x + 6}{x^2 - 4}$

a) $\frac{ax + ay}{az} = \frac{\not{a}(x + y)}{\not{a}z} = \frac{x + y}{z}$

b) $\frac{3x + 6}{x^2 - 4} = \frac{3(x+2)}{(x+2)(x-2)} = \frac{3}{x-2}$

Watch out for the difference of two squares.

2) If there's a **fraction** in the numerator or denominator (e.g. $\frac{1}{x}$), **multiply** the **whole thing** (i.e. top and bottom) by the same factor to get **rid of it** (e.g. for $\frac{1}{x}$, you'd multiply through by x).

Example: Simplify $\frac{2 + \frac{1}{2x}}{4x^2 + x}$

$$\frac{2 + \frac{1}{2x}}{4x^2 + x} = \frac{\left(2 + \frac{1}{2x}\right) \times 2x}{x(4x + 1) \times 2x} = \frac{4x + 1}{2x^2(4x + 1)} = \frac{1}{2x^2}$$

3) You **multiply** algebraic fractions in exactly the same way as normal fractions — multiply the **numerators** together, then multiply the **denominators**. It's a good idea to **cancel** any **common factors** before you multiply.

4) To **divide** by an algebraic fraction, you just **multiply** by its **reciprocal** (the reciprocal is 1 ÷ the original thing — for fractions you just turn the fraction **upside down**).

Examples: Simplify the following: a) $\frac{x^2 - 2x - 15}{2x + 8} \times \frac{x^2 - 16}{x^2 + 3x}$ b) $\frac{3x}{5} \div \frac{3x^2 - 9x}{20}$

a) $\frac{x^2 - 2x - 15}{2x + 8} \times \frac{x^2 - 16}{x^2 + 3x} = \frac{(x+3)(x-5)}{2(x+4)} \times \frac{(x+4)(x-4)}{x(x+3)}$

$= \frac{(x-5)(x-4)}{2x} \quad \left(= \frac{x^2 - 9x + 20}{2x}\right)$

Factorise both fractions.

Turn the second fraction upside down.

b) $\frac{3x}{5} \div \frac{3x^2 - 9x}{20} = \frac{3x}{5} \times \frac{20}{3x(x-3)}$

$= \frac{4}{x-3}$

Polynomials

Add *and* ***Subtract*** *fractions by finding a* ***Common Denominator***

You'll have come across **adding** and **subtracting fractions** before, so here's a little reminder of how to do it:

Example: Simplify $\frac{2y}{x(x+3)} + \frac{1}{y^2(x+3)} - \frac{x}{y}$

1) Find the common denominator

Take all the individual 'bits' from the bottom lines and multiply them together. Only use each bit once unless something on the bottom line is raised to a power:

The individual 'bits' here are x, (x + 3) and y. But you need to use y^2 because there's a y^2 in the second fraction's denominator.

common denominator = $xy^2(x+3)$

2) Put each fraction over the common denominator

Make the denominator of each fraction into the common denominator. Multiply the top and bottom lines of each fraction by whatever makes the bottom line the same as the common denominator:

$$\frac{y^2 \times 2y}{y^2x(x+3)} + \frac{x \times 1}{xy^2(x+3)} - \frac{xy(x+3) \times x}{xy(x+3)y}$$

3) Combine into one fraction

Once everything's over the common denominator you can just add the top lines together and simplify the numerator:

$$= \frac{2y^3 + x - x^2y(x+3)}{xy^2(x+3)} = \frac{2y^3 + x - x^3y - 3x^2y}{xy^2(x+3)}$$

All the bottom lines are the same — so you can just add the top lines.

Practice Questions

Q1 Expand the brackets and simplify the following expressions:

a) $(x+y)(x-y)$

b) $(x+y)(x+y)$

c) $35xy + 25y(5y + 7x) - 100y^2$

d) $(x + 3y + 2)(3x + y + 7)$

Q2 Show that $(\sqrt{x} + \sqrt{2})(\sqrt{x} - \sqrt{2}) = x - 2$.

Q3 Take out the common factors from the following expressions:

a) $2x^2y + axy + 2xy^2$

b) $a^2x + a^2b^2x^2$

c) $16y + 8yx + 56x$

d) $x(x-2) + 3(2-x)$

Q4 Factorise the following quadratics:

a) $x^2 + 6x - 7$

b) $x^2 - 4x - 12$

c) $9x^2 - 64$

d) $4x^2 - 11x - 20$

Q5 Simplify the following:

a) $\frac{4x^2 - 25}{6x - 15}$

b) $\frac{2x+3}{x-2} \times \frac{4x-8}{2x^2-3x-9}$

c) $\frac{x^2-3x}{x+1} \div \frac{x}{2}$

Exam Questions

Q1 Write $\frac{2x^2 - 9x - 35}{x^2 - 49}$ as a fraction in its simplest form. [3 marks]

Q2 Factorise $2x^4 - 32x^2$ completely. [2 marks]

Q3 Write each of the following polynomials as a single fraction in its simplest form.

a) $\frac{x}{2x+1} + \frac{3}{x^2} + \frac{1}{x}$ [3 marks]

b) $\frac{2}{x^2-1} - \frac{3x}{x-1} + \frac{x}{x+1}$ [3 marks]

What do you call a hungry parrot? Polynomials...

Nothing on these pages should be a big shock to you — you've been using normal fractions for years, and algebraic fractions work in just the same way. They look a bit scary, but they're all warm and fuzzy inside.

Algebraic Division

I'm going to spoil you with three methods for algebraic division — these can be a bit tricky so take your time with them. I know you can't wait to get stuck into them, but first you need to get to know the Factor Theorem.

*There are some **Terms** you need to **Know***

These words will keep popping up over the next few pages, so make sure you know what they all mean.

1) **DEGREE** — the highest power of x in the polynomial (e.g. the degree of $4x^5 + 6x^2 - 3x - 1$ is 5).
2) **DIVISOR** — this is the thing you're dividing by (e.g. if you divide $x^2 + 4x - 3$ by $x + 2$, the divisor is $x + 2$).
3) **QUOTIENT** — the stuff that you get when you divide by the divisor (not including the remainder).
4) **REMAINDER** — the bit that's left over at the end (for A-level maths this will be a constant).

*The **Factor Theorem** links **Roots** and **Factors***

If you get a **remainder of zero** when you divide the polynomial f(x) by ($x - a$), then ($x - a$) must be a factor of f(x). That's the **Factor Theorem**:

If $f(x)$ is a polynomial, and $f(a) = 0$, then $(x - a)$ is a factor of $f(x)$.

If $f\left(\frac{b}{a}\right) = 0$, then $(ax - b)$ is a factor of $f(x)$.

In other words: If you know the roots, you also know the factors — and vice versa.

Example: Show that $(2x + 1)$ is a factor of $f(x) = 2x^3 - 3x^2 + 4x + 3$.

Use the second version of the Factor Theorem — in this case, $a = 2$ and $b = -1$.

This means that if you show that $f\left(-\frac{1}{2}\right) = 0$, then, by the Factor Theorem, $(2x + 1)$ is a factor.

$$f(x) = 2x^3 - 3x^2 + 4x + 3 \text{ and so } f\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{8}\right) - 3\left(\frac{1}{4}\right) + 4\left(-\frac{1}{2}\right) + 3 = 0$$

So, by the Factor Theorem, $(2x + 1)$ is a factor of f(x).

Example: The quartic polynomial $g(x) = x^4 - 8x^3 - 87x^2 + 342x + 1512$ has roots at $x = 12, 6, -3$ and -7. Fully factorise g(x).

If a is a root of g(x) then $g(a) = 0$. So $(x - a)$ is a factor of g(x). Since you're given four roots, you have four factors and that's all you need to fully factorise a quartic. Hence:

$$g(x) = x^4 - 8x^3 - 87x^2 + 342x + 1512 = (x - 12)(x - 6)(x + 3)(x + 7).$$

You could check that this factorisation is correct by expanding the brackets back out.

Method 1** — **Divide** by **Subtracting Multiples** of the **Divisor

This is the first of **three methods** I'm going to show you for dividing a polynomial by a **linear expression**. To do **algebraic division** you can keep **subtracting** chunks of the **divisor**, ($x - k$), until you get the **remainder**.

Algebraic Division

1. Subtract a multiple of $(x - k)$ to get rid of the highest power of x.
2. Repeat step 1 until you've got rid of all the powers of x.
3. Work out how many lumps of $(x - k)$, you've subtracted, and read off the remainder.

For this course you'll only have to divide by a linear expression — i.e. ax + b, where a and b are constants. This means that the remainder will always be a constant because the degree of the remainder has to be less than the degree of the divisor.

*The **Remainder Theorem** is an easy way to work out **Remainders***

If $f(x)$ is a **polynomial** then the **Remainder Theorem** states that:

When you divide $\mathbf{f(x)}$ by $\mathbf{(x - a)}$, the remainder is $\mathbf{f(a)}$.

When you divide $\mathbf{f(x)}$ by $\mathbf{(ax - b)}$, the remainder is $\mathbf{f\left(\frac{b}{a}\right)}$.

Example: Find the remainder when you divide $2x^3 - 3x^2 - 3x + 7$ by $2x - 1$.

$f(x) = 2x^3 - 3x^2 - 3x + 7$. The **divisor** is $2x - 1$, so $a = 2$ and $b = 1$.

Using the Remainder Theorem, the **remainder** must be $f\left(\frac{1}{2}\right) = 2\left(\frac{1}{8}\right) - 3\left(\frac{1}{4}\right) - 3\left(\frac{1}{2}\right) + 7 = 5$

Algebraic Division

*Always get **Rid** of the **Highest Power** of x*

Example: Divide $(2x^3 - 3x^2 - 3x + 7)$ by $(x - 2)$

You're asked to calculate $(2x^3 - 3x^2 - 3x + 7) \div (x - 2)$. Start with $2x^3 - 3x^2 - 3x + 7$, and subtract $2x^2$ lots of $(x - 2)$ to get rid of the x^3 term:

$$(2x^3 - 3x^2 - 3x + 7) - 2x^2(x - 2)$$
$$= (2x^3 - 3x^2 - 3x + 7) - 2x^3 + 4x^2$$
$$= x^2 - 3x + 7$$

This is what's left — so now you have to get rid of the x^2 term.

Now start again with $x^2 - 3x + 7$. The highest power of x is the x^2 term. So subtract x lots of $(x - 2)$ to get rid of that:

$$(x^2 - 3x + 7) - x(x - 2)$$
$$= (x^2 - 3x + 7) - x^2 + 2x$$
$$= -x + 7$$

All that's left now is $-x + 7$. Get rid of the $-x$ by subtracting -1 lots of $(x - 2)$:

$$(-x + 7) - (-1(x - 2))$$
$$= (-x + 7) + x - 2 = 5$$

So $(2x^3 - 3x^2 - 3x + 7) \div (x - 2) = $ **$2x^2 + x - 1$ remainder 5**

Method 2** — use **Algebraic Long Division

To divide two **algebraic** expressions, you can use **long division** (using the same method you'd use for numbers).

Example: Divide $2x^3 - 7x^2 - 16x + 11$ by $x - 5$.

If the original polynomial doesn't have an x term, for example, just put 0x where the x term should be.

$$\begin{array}{rl}
 & 2x^2 + 3x - 1 \\
x - 5 \,\big)\!\!\! & 2x^3 - 7x^2 - 16x + 11 \\
- & (2x^3 - 10x^2) \\
 & 3x^2 - 16x \\
- & (3x^2 - 15x) \\
 & -x + 11 \\
- & (-x + 5) \\
 & 6
\end{array}$$

① $2x^3 \div x = 2x^2$

② Multiply $(x - 5)$ by $2x^2$ to get this.

③ Subtracting gives $3x^2$, so divide this by x to get 3x.

④ Multiply $(x - 5)$ by 3x to get this, then subtract again.

⑤ Divide $-x$ by x to get -1, then multiply $(x - 5)$ by -1.

⑥ After subtracting, this term has a degree that's less than the degree of the divisor, $(x - 5)$, so it can't be divided. This is the remainder.

These two terms haven't changed — they've just dropped down to make the subtraction clearer.

So $(2x^3 - 7x^2 - 16x + 11) \div (x - 5) = $ **$2x^2 + 3x - 1$ remainder 6**.

This could also be written as $\frac{2x^3 - 7x^2 - 16x + 11}{x - 5} = 2x^2 + 3x - 1 + \frac{6}{x - 5}$.

You can multiply your answer by $(x - 5)$ to check you've got it right.

Method 3** — use the **Formula f(x) = q(x)d(x) + r(x)

There's a **formula** you can use to do **algebraic division** — it looks like this:

> A polynomial f(x) can be written in the form $f(x) \equiv q(x)d(x) + r(x)$, where q(x) is the quotient, d(x) is the divisor and r(x) is the remainder.

You'll be given f(x) and d(x) in the **question**, and it's down to you to **work out** q(x) and r(x). Here's how you do it:

Using the Formula

① First, you have to work out the **degrees** of the **quotient** and **remainder**, which depend on the degrees of the polynomial and the divisor. The degree of the quotient is **deg f(x) – deg d(x)**, and the degree of the remainder, **deg r(x)**, has to be less than the degree of the divisor.

② Write out the division using the **formula** above, but replace q(x) and r(x) with **general polynomials** (i.e. a general polynomial of degree 2 is $Ax^2 + Bx + C$, and a general polynomial of degree 1 is $Ax + B$, where A, B, C, etc. are constants to be found).

③ The next step is to work out the values of the **constants** — you do this by **substituting** in values for x to make bits disappear, and by **equating coefficients**. It's best to start with the constant term and work backwards from there.

④ Finally, **write out** the division again, replacing A, B, C, etc. with the values you've found.

For this course, you'll only get questions where deg d(x) = 1 and deg r(x) = 0.

Equating coefficients means comparing the coefficients of each power of x on the LHS and the RHS.

The method looks a bit **intense**, but follow through the **example** on the next page to see how it works.

Algebraic Division

Start with the *Remainder* and *Work Backwards*

When you're using this method, you might have to use **simultaneous equations** to work out some of the coefficients.

Example: Divide $x^4 - 3x^3 - 3x^2 + 10x + 5$ by $x - 2$.

① The polynomial f(x) has degree 4 and the divisor d(x) has degree 1, which means that the quotient q(x) has degree 4 – 1 = 3 (i.e. a **cubic**). The remainder r(x) has degree 0.

② Write out the division in the form $f(x) \equiv q(x)d(x) + r(x)$:
$x^4 - 3x^3 - 3x^2 + 10x + 5 \equiv (Ax^3 + Bx^2 + Cx + D)(x - 2) + E$

Remember, it's easiest to start by finding the constant term.

③ **Substitute $x = 2$** into the identity to make the q(x)d(x) bit disappear. This gives the remainder as **E = 5**.
Now, using this value of E and putting $x = 0$ into the identity gives the equation 5 = –2D + 5, so **D = 0**.
Using the values of D and E you now have:

$$x^4 - 3x^3 - 3x^2 + 10x + 5 \equiv (Ax^3 + Bx^2 + Cx)(x - 2) + 5$$
$$\equiv Ax^4 + (B - 2A)x^3 + (C - 2B)x^2 - 2Cx + 5$$

Equating coefficients of x^4, x^3 and x gives **A = 1**, B – 2A = –3 (so **B = –1**) and –2C = 10 (so **C = –5**).

④ Putting these values into the original identity gives: $x^4 - 3x^3 - 3x^2 + 10x + 5 \equiv (x^3 - x^2 - 5x)(x - 2) + 5$.
So the answer is **$(x^3 - x^2 - 5x)$ remainder 5**.

Practice Questions

Q1 The polynomial $f(x) = x^4 - 3x^3 + 7x^2 - 12x + 14$ is divided by $x + 2$.
a) What is the divisor, d(x)?
b) What is the value of: (i) the degree of d(x)? (ii) the degree of the quotient, q(x)?

Q2 Which of the following are factors of $f(x) = x^5 - 4x^4 + 3x^3 + 2x^2 - 2$?
a) $x - 1$ b) $x + 1$ c) $x - 2$ d) $2x - 2$

Q3 Use algebraic long division to divide $x^3 + 2x^2 - x + 19$ by $x + 4$.

Q4 Write the following functions f(x) in the form $f(x) = (x + 2)q(x) + r(x)$, where q($x$) is a quadratic:
a) $f(x) = 3x^3 - 4x^2 - 5x - 6$, b) $f(x) = x^3 + 2x^2 - 3x + 4$, c) $f(x) = 2x^3 + 6x - 3$

Q5 Write $2x^3 + 8x^2 + 7x + 8$ in the form $(Ax^2 + Bx + C)(x + 3) + D$.
Using your answer, state the result when $2x^3 + 8x^2 + 7x + 8$ is divided by $(x + 3)$.

Exam Questions

Q1 a) State the Factor Theorem for a polynomial f(x). [1 mark]
b) Hence, determine whether the following are factors of $f(x) = 2x^3 - 5x^2 - 4x + 3$.
(i) $(x - 1)$ [1 mark]
(ii) $(x + 1)$ [1 mark]
c) Factorise f(x) completely. [3 marks]

Q2 $f(x) = (4x^2 + 3x + 1)(x - p) + 5$, where p is a constant.
a) State the value of f(p). [1 mark]
b) Find the value of p, given that when f(x) is divided by $(x + 1)$, the remainder is 0. [2 marks]
c) Determine whether $(x - 1)$ is a factor of f(x). [1 mark]

Q3 Write $x^3 + 15x^2 + 43x - 30$ in the form $(Ax^2 + Bx + C)(x + 6) + D$,
where A, B, C and D are constants to be found. [3 marks]

Just keep repeating — divide and conquer, divide and conquer...

There's a lot to take in about algebraic division so feel free to go over it again. It's up to you which method you prefer but I'd recommend either long division or the formula — these are a bit quicker than the first method. And if you think a divisor might actually be a factor, just use the Factor Theorem instead of actually dividing.