Cubics

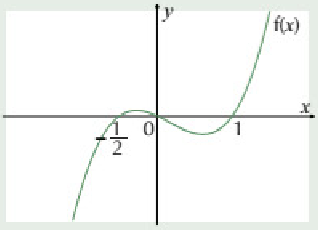
If you know the *Factors* of a cubic — the graph's easy to *Sketch*

All cubics have a similar shape: '**bottom-left to top-right**' if the coefficient of x^3 is **positive** or '**top-left to bottom-right**' if the coefficient of x^3 is **negative**.

Once you know the **factors** of a cubic, the graph is easy to sketch — just find where the function is **zero**.

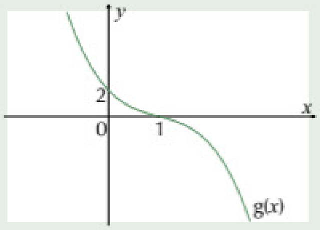
Example: Sketch the graphs of the following cubic functions:

a) $f(x) = x(x-1)(2x+1)$



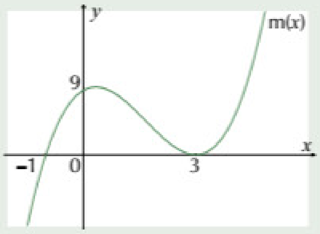
The function's zero when x = 0, 1 or $-\frac{1}{2}$.

b) $g(x) = (1-x)(x^2-2x+2)$



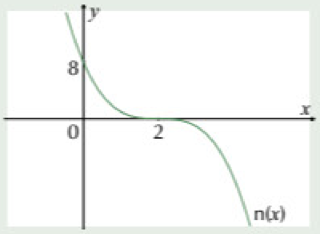
The coefficient of x^3 is negative, and the quadratic factor of g(x) has no real roots — so g(x) is only zero once.

c) $m(x) = (x-3)^2(x+1)$



This has a 'double root' at x = 3, so the graph just touches the x-axis there but doesn't go through.

d) $n(x) = (2-x)^3$



This has a 'triple root' at x = 2, and the coefficient of x^3 is negative.

Peter spent too much time sketching his cubic graph in the exam.

The **power** of a factor $(x-a)^n$ affects what happens at $x = a$:
even power ⇒ the curve **touches** the x-axis but doesn't cross it (e.g. the 'double root' in m(x) above)
odd power ⇒ the curve **crosses** the x-axis (e.g. the 'triple root' in n(x) above).

This is also true for other polynomials — e.g. quartics (p.30).

Practice Questions

Q1 Sketch these cubic graphs. Go on, this is the fun part.
a) $y = (x-4)^3$ b) $y = (3-x)(x+2)^2$ c) $y = (1-x)(x^2-6x+8)$ d) $y = (x-1)(x-2)(x-3)$

Q2 Show that the expressions below are factors of the given functions. Hence factorise the functions fully.
a) $(x-1)$ is a factor of $f(x) = x^3 - x^2 - 2x + 2$ b) $(x+4)$ is a factor of $g(x) = x^3 + 3x^2 - 10x - 24$
c) $(2x-1)$ is a factor of $h(x) = 2x^3 + 3x^2 - 8x + 3$ d) $(3x-2)$ is a factor of $k(x) = 3x^3 + 10x^2 + 10x - 12$

Q3 Given that $(x+5)$ is a factor of $f(x) = x^3 - 3x^2 - 33x + 35$, factorise f(x) fully.

Exam Questions

Q1 The curve C has the equation $y = (2x+1)(x-2)^2$
Sketch C, clearly showing the points at which the curve meets the x- and y-axes [3 marks]

Q2 a) Show that $(2x+1)$ is a factor of $f(x) = 6x^3 + 37x^2 + 5x - 6$ [2 marks]
b) Hence, or otherwise, factorise f(x) fully. [2 marks]
c) Sketch $y = f(x)$, clearly showing the points at which the curve meets the x- and y-axes. [3 marks]

Q3 If $f(x) = 7x^3 - 26x^2 + 13x + 6$, solve $f(x) = 0$. [4 marks]

Does your cubic have the x factor? Only if x is in every term...

Factorising cubics might seem daunting but once you're used to the method it gets easier. For sketches, always find where the cubic crosses the x- and y-axis — even if you're not sure what shape the curve will be. This goes for most other kinds of graph too, not just cubics. You'll be seeing some of these other curves very soon — happy sketching.

Graphs of Functions

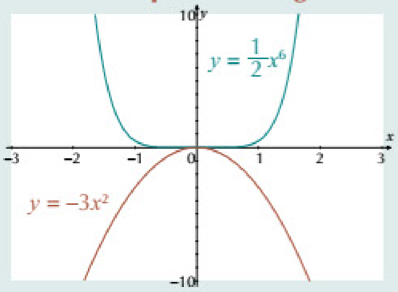
A picture speaks a thousand words... and graphs are as close as you're going to get in maths. They're dead useful for getting your head round tricky questions, and time spent learning how to sketch graphs is time well spent.

The graph of $y = kx^n$ *is a different* ***Shape*** *for different* ***k*** *and* ***n***

Usually, you only need a **rough** sketch of a graph — so just knowing the basic shapes of these graphs will do.

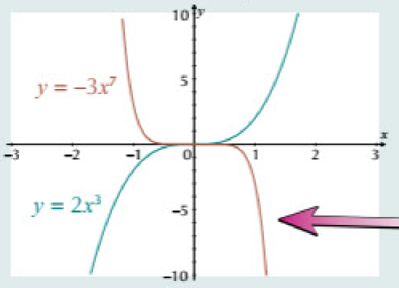
1 n positive and even

You get a **u-shape** (if k is **positive**) or an **n-shape** (if k is **negative**).



2 n positive and odd

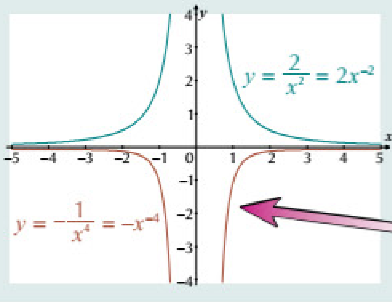
You get a '**corner-to-corner**' shape.



If k is negative, you get a 'top-left to bottom-right' shape.

3 n negative and even

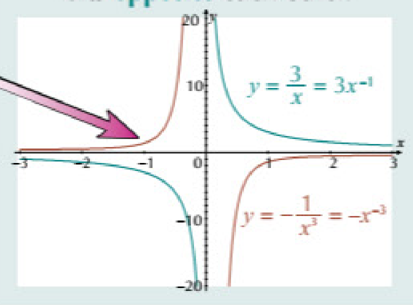
You get a graph with **two** bits **next to** each other.



If k is negative, the graph is below the x-axis.

4 n negative and odd

You get a graph with **two** bits **opposite** each other.



If k is negative, it's in the bottom-right and the top-left quadrants.

An **asymptote** of a curve is a **line** which the curve gets **infinitely close** to, but **never touches**.
So graphs 3 and 4 both have asymptotes at $x = 0$ and $y = 0$.

Find where the curve ***Crosses*** *the* ***x-axis*** *to sketch* ***Quartics***

A **quartic** has an x^4 term as the highest power. If you're asked to sketch one of these it's likely to be **factorised**, so you can easily work out where it **crosses** or **touches** the x-axis. Then you can figure out what the curve looks like.

Quartics with **positive coefficients** of x^4 are always positive for **very** positive and negative values of x. For **negative coefficients** of x^4 the curve is negative for **very** positive and negative x-values — this is similar to quadratics.

Example: Sketch the graph of $f(x) = x(x + 1)(x - 2)^2$.

The coefficient of x^4 is positive so the graph is positive for very positive and negative x-values.

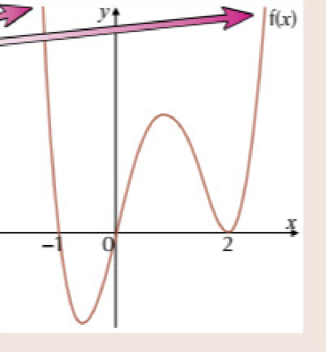
$$x(x + 1)(x - 2)^2 = 0$$

The curve crosses the x-axis at x = 0.

The curve crosses the x-axis at x = –1.

This is a double root, so the curve touches the x-axis at x = 2, but doesn't cross it.

Substituting in x = 0 gives y = 0, so the curve crosses the y-axis at y = 0.



If there was a triple root, the graph would cross the x-axis, but flatten out at the same time — like it does in the middle of $y = x^3$ (or in n(x) in the example on p.27).

Proportion

Variables that are in proportion are closely related. Think of this page as like a daytime TV DNA test for variables.

Direct Proportion** graphs are **Straight Lines** through the **Origin

If two variables are in **direct proportion**, it means that changing one variable will change the other by the same scale factor. So multiplying or dividing by **any** constant will have the same effect on both variables.
To say that "y is directly proportional to x", you can write:

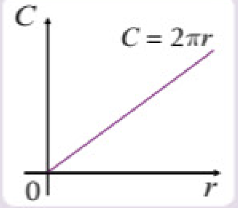
$y \propto x$ which is equivalent to writing $y = kx$ ← k is sometimes called the constant of proportionality.

Example: The circumference of a circle, C, is directly proportional to its radius, r.
a) Find the constant of proportionality and sketch the graph of C against r.
b) A circle with a radius of p cm has a circumference of 13 cm.
Find the circumference of a circle with a radius of $2.5p$ cm.

y = kx is a straight line with gradient k that passes through the origin (0, 0).

a) The circumference of a circle is given by $2\pi r$:
$C \propto r$ means $C = kr$. So $kr = 2\pi r \Rightarrow$ $\boldsymbol{k = 2\pi}$

The graph is a straight line through the origin with gradient 2π:



b) You can do this without using the circumference formula. The radius of the second circle is 2.5 times the size of the first so the circumference will be 2.5 times the first as well:
$C = 2.5 \times 13 =$ **32.5 cm**

Inverse Proportion** graphs are of the form **y = k / x

If two variables are in **inverse proportion**, it means that changing one variable will change the other by the **reciprocal** of the scale factor. So **multiplying** one variable by **any** constant is the same as **dividing** the other by the same constant.
Saying that "y is inversely proportional to x" is the same as saying "y is directly proportional to $\frac{1}{x}$", so you can write:

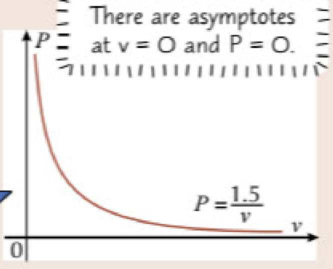
$y \propto \frac{1}{x}$ which is equivalent to writing $y = \frac{k}{x}$ ← k is still the constant of proportionality.

Example: The pressure of a gas, P N/m², is modelled as being inversely proportional to the volume of its container, v m³.
a) A container with volume 12 m³ contains a gas with pressure 0.125 N/m². Find the constant of proportionality.
b) Sketch the graph of P against v.

a) $P \propto \frac{1}{v}$ is the same as saying $P = \frac{k}{v}$.
Use the values from the question:
When $v = 12$, $P = 0.125$, so $0.125 = \frac{k}{12}$
$\Rightarrow$ $\boldsymbol{k} = 0.125 \times 12 =$ **1.5**

b) $P = \frac{1.5}{v}$ is of the form $P = kv^n$ with $k = 1.5$ and $n = -1$ (see p.30).
But volume cannot be negative so you only need positive values of v.

There are asymptotes at v = 0 and P = 0.



Practice Questions

Q1 If m is directly proportional to n and $m = 12$ when $n = 3$, find the constant of proportionality, k.
Q2 If p is inversely proportional to q and $p = 3$ when $q = 5$, find the constant of proportionality, k.

Exam Question

Q1 After a storm, the area of a small island was reduced by erosion in the following years. The area, A km², of the island is modelled as being inversely proportional to t, the time in years since the storm, for $t \geq 1$.
a) 5.5 years after the storm, the area of the island was 2.6 km². Find the constant of proportionality, k. [1 mark]
b) Sketch the graph of t against A, stating the equations of any asymptotes. [2 marks]
c) Suggest one reason why this model has the restriction $t \geq 1$. [1 mark]

Time spent checking your phone is inversely proportional to exam marks...

Watch out for other proportion relationships — e.g. you might come across relations such as $y \propto x^2$ or $y \propto 1/\sqrt{x}$.