Geometry of Lines and Circles

You need to know about a billion things about lines, midpoints, circles, equations — all manner of geometrical goodies. Okay, maybe not quite a billion, but enough to fill four whole pages, and that's more than enough for me.

*Finding the **Equation of a Line***

You need to know three different ways of writing the equation of a straight line:

$y - y_1 = m(x - x_1)$ $\quad$ $y = mx + c$ $\quad$ $ax + by + c = 0$ where a, b and c are **integers**

You might be asked to write the equation of a line in **any** of these forms — but they're all similar. Basically, if you find an equation in one form, you can easily **convert** it into either of the others.

*The Easiest to find is **$y - y_1 = m(x - x_1)$**...*

Equations of Lines

1) LABEL the points (x_1, y_1) and (x_2, y_2).
2) GRADIENT — find it and call it m.
3) WRITE DOWN THE EQUATION using $y - y_1 = m(x - x_1)$.
4) CONVERT to one of the other forms, if necessary.

Example: Find the equation of the line that passes through the points (–3, 10) and (1, 4), in the form $y - y_1 = m(x - x_1)$.

Label the points: $(x_1, y_1) = (-3, 10)$ and $(x_2, y_2) = (1, 4)$

It doesn't matter which way round you label them.

Find m, the **gradient** of the line: $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 10}{1 - (-3)} = \frac{-6}{4} = -\frac{3}{2}$

Be careful here — y goes on the top, x on the bottom.

Write down the equation of the line:

$y - y_1 = m(x - x_1)$

$y - 10 = -\frac{3}{2}(x - (-3))$ ← $x_1 = -3$ and $y_1 = 10$

$\mathbf{y - 10 = -\frac{3}{2}(x + 3)}$

You might recognise this method for finding m from GCSE.

*...and then you can **Rearrange***

Once you've got the equation in the form $y - y_1 = m(x - x_1)$, it's pretty easy to **convert** it to either of the other forms. Here's how you'd do it for the example above:

For the form $y = mx + c$, take **everything except the y** over to the right.

$y - 10 = -\frac{3}{2}(x + 3)$

$\Rightarrow \quad y = -\frac{3}{2}x - \frac{9}{2} + 10$

$\Rightarrow \quad \mathbf{y = -\frac{3}{2}x + \frac{11}{2}}$

To find the form $ax + by + c = 0$, take **everything** over to one side — and then get rid of any fractions.

$y = -\frac{3}{2}x + \frac{11}{2}$

$\Rightarrow \quad \frac{3}{2}x + y - \frac{11}{2} = 0$

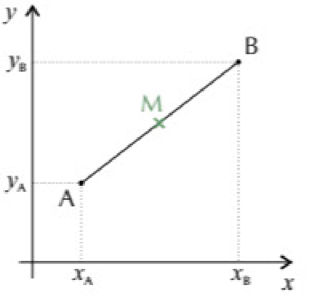
$\Rightarrow \quad \mathbf{3x + 2y - 11 = 0}$

a, b and c have to be integers, so multiply the whole equation by 2 to get rid of the 2s on the bottom of the fractions.

If you end up with an equation like $\frac{3}{2}x - \frac{4}{3}y + 6 = 0$, where you've got a 2 and a 3 on the bottom of the fractions, multiply everything by the lowest common multiple of 2 and 3, i.e. 6.

*You can find the **Midpoint** of a **Line Segment***

A **line segment** is just a straight line that goes between two points. Since it has a beginning and an end, it also has a middle (like all good books), and you can work out the coordinates of the **midpoint** using the formula:



$$\text{Midpoint (AB)} = \left(\frac{x_A + x_B}{2}, \frac{y_A + y_B}{2}\right)$$

You could think of this formula as the mean of the x-coordinates and the mean of the y-coordinates.

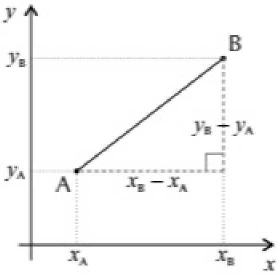
Example: Find the midpoint of the line segment between (4, 3) and (–2, 5).

Use the midpoint formula: $\quad \text{Midpoint} = \left(\frac{4 + (-2)}{2}, \frac{3 + 5}{2}\right) = \left(\frac{2}{2}, \frac{8}{2}\right) = \mathbf{(1, 4)}$

Geometry of Lines and Circles

Use *Pythagoras' Theorem* to find the *Length* of a line segment

To find the **length** of a line segment (or the **distance between two points**), you can imagine a right-angled triangle, like in the diagram on the right. Then it's just a matter of using Pythagoras' theorem:



$$\text{Length (AB)} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$$

Example: Find the exact length of the segment of the line with equation $y = 5x - 2$ between $x = 3$ and $x = 4$.

First, find the y-coordinates at $x = 3$ and $x = 4$:

When $x = 3$, $y = 5(3) - 2 \Rightarrow y = 15 - 2 = 13$

When $x = 4$, $y = 5(4) - 2 \Rightarrow y = 20 - 2 = 18$

So: $(x_A, y_A) = (3, 13)$ and $(x_B, y_B) = (4, 18)$

Now substitute these values into the formula:

$$\text{Length} = \sqrt{(4-3)^2 + (18-13)^2} = \sqrt{1+25} = \mathbf{\sqrt{26}}$$

The question says "exact", so leave it as a surd.

Pythagoras had to go to great lengths to discover his theorem.

Parallel Lines have equal *Gradient*

That's what makes them parallel — the fact that the gradients are the same.

Example: The equation of the line l_1 can be written as $y = \frac{3}{4}x - \frac{7}{4}$ or $3x - 4y - 7 = 0$. Find the line parallel to l_1 that passes through the point (3, –1).

Parallel lines have the same gradient.

The original equation is: $y = \frac{3}{4}x - \frac{7}{4}$

So the new equation will be: $y = \frac{3}{4}x + c$ ← We just need to find c.

We know that the line passes through (3, –1), so stick x = 3, y = –1 into the equation to find c.

At (3, –1), $-1 = \frac{3}{4} \times 3 + c \Rightarrow c = -1 - \frac{9}{4} = -\frac{13}{4}$

So the equation of the line is: $\mathbf{y = \frac{3}{4}x - \frac{13}{4}}$

Or with the ax + by + c = 0 form it's even easier:

The original line is: $3x - 4y - 7 = 0$

So the new line is: $3x - 4y + k = 0$

Then use the values of x and y at (3, –1) to find k.

$3 \times 3 - 4 \times (-1) + k = 0$

$\Rightarrow 9 + 4 + k = 0$

$\Rightarrow k = -13$

So the equation is: $\mathbf{3x - 4y - 13 = 0}$

The gradient of a *Perpendicular* line is *–1 ÷ the Other Gradient*

Finding **perpendicular** lines (or '**normals**') is just as easy as finding parallel lines — as long as you remember the gradient of the perpendicular line is **–1 ÷ the gradient of the other one**.

Example: The equation of the line l_2 can be written as $y = \frac{1}{3}x - 1$ or $x - 3y - 3 = 0$. Find the line perpendicular to l_2 that passes through the point (–2, 4).

l_2 has equation: $y = \frac{1}{3}x - 1$

So if the equation of the new line is $y = mx + c$, then

$m = -1 \div \frac{1}{3}$

$\Rightarrow m = -3$

Since the gradient of a perpendicular line is –1 ÷ the other one.

At $x = -2$, $y = 4$: $4 = (-3) \times (-2) + c$

$\Rightarrow c = 4 - 6 = -2$

So the equation of the perpendicular line is $\mathbf{y = -3x - 2}$

Or if you start with: l_2 $x - 3y - 3 = 0$

To find a perpendicular line, swap these two numbers around, and change the sign of **one of them**. So here, 1 and –3 become 3 and 1.

So the new line has equation $3x + y + k = 0$

Or you could have used $-3x - y + k = 0$.

So at $x = -2$, $y = 4$: $3 \times (-2) + 4 + k = 0$

$\Rightarrow k = 2$

And so the equation of the perpendicular line is $\mathbf{3x + y + 2 = 0}$

Proportion

Variables that are in proportion are closely related. Think of this page as like a daytime TV DNA test for variables.

Direct Proportion graphs are Straight Lines through the Origin

If two variables are in **direct proportion**, it means that changing one variable will change the other by the same scale factor. So multiplying or dividing by **any** constant will have the same effect on both variables.
To say that "y is directly proportional to x", you can write:

$y \propto x$ which is equivalent to writing $y = kx$

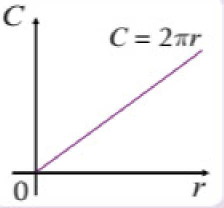
k is sometimes called the constant of proportionality.

Example: The circumference of a circle, C, is directly proportional to its radius, r.
a) Find the constant of proportionality and sketch the graph of C against r.
b) A circle with a radius of p cm has a circumference of 13 cm. Find the circumference of a circle with a radius of $2.5p$ cm.

$y = kx$ is a straight line with gradient k that passes through the origin (0, 0).

a) The circumference of a circle is given by $2\pi r$:
$C \propto r$ means $C = kr$. So $kr = 2\pi r \Rightarrow$ $\boldsymbol{k = 2\pi}$

The graph is a straight line through the origin with gradient 2π:



b) You can do this without using the circumference formula. The radius of the second circle is 2.5 times the size of the first so the circumference will be 2.5 times the first as well:
$C = 2.5 \times 13 =$ **32.5 cm**

Inverse Proportion graphs are of the form y = k / x

If two variables are in **inverse proportion**, it means that changing one variable will change the other by the **reciprocal** of the scale factor. So **multiplying** one variable by **any** constant is the same as **dividing** the other by the same constant.
Saying that "y is inversely proportional to x" is the same as saying "y is directly proportional to $\frac{1}{x}$", so you can write:

$y \propto \frac{1}{x}$ which is equivalent to writing $y = \frac{k}{x}$

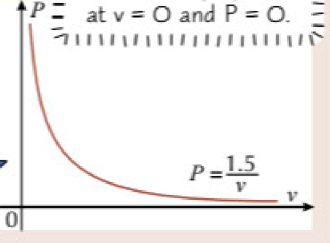
k is still the constant of proportionality.

Example: The pressure of a gas, P N/m², is modelled as being inversely proportional to the volume of its container, v m³.
a) A container with volume 12 m³ contains a gas with pressure 0.125 N/m². Find the constant of proportionality.
b) Sketch the graph of P against v.

a) $P \propto \frac{1}{v}$ is the same as saying $P = \frac{k}{v}$.
Use the values from the question:
When $v = 12$, $P = 0.125$, so $0.125 = \frac{k}{12}$
$\Rightarrow$ $k = 0.125 \times 12 =$ **1.5**

b) $P = \frac{1.5}{v}$ is of the form $P = kv^n$ with $k = 1.5$ and $n = -1$ (see p.30). But volume cannot be negative so you only need positive values of v.

There are asymptotes at v = 0 and P = 0.



Practice Questions

Q1 If m is directly proportional to n and $m = 12$ when $n = 3$, find the constant of proportionality, k.
Q2 If p is inversely proportional to q and $p = 3$ when $q = 5$, find the constant of proportionality, k.

Exam Question

Q1 After a storm, the area of a small island was reduced by erosion in the following years. The area, A km², of the island is modelled as being inversely proportional to t, the time in years since the storm, for $t \geq 1$.
a) 5.5 years after the storm, the area of the island was 2.6 km². Find the constant of proportionality, k. [1 mark]
b) Sketch the graph of t against A, stating the equations of any asymptotes. [2 marks]
c) Suggest one reason why this model has the restriction $t \geq 1$. [1 mark]

Time spent checking your phone is inversely proportional to exam marks...

Watch out for other proportion relationships — e.g. you might come across relations such as $y \propto x^2$ or $y \propto 1/\sqrt{x}$.