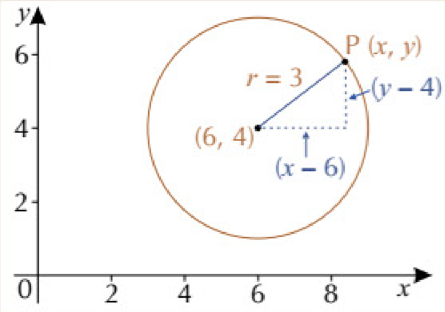
Geometry of Lines and Circles

Equation of a circle: $(x - a)^2 + (y - b)^2 = r^2$

The equation of a circle looks complicated, but it's all based on Pythagoras' theorem.
Take a look at the circle below, with centre (6, 4) and radius 3.



Joining a point P (x, y) on the circumference of the circle to its centre (6, 4), we can create a **right-angled triangle**.

Now let's see what happens if we use **Pythagoras' theorem**:

$$(x - 6)^2 + (y - 4)^2 = 3^2$$

or: $(x - 6)^2 + (y - 4)^2 = 9$ ← This is the equation for the circle. It's as easy as that.

In general, a circle with radius r and centre (a, b) has the equation:

$$(x - a)^2 + (y - b)^2 = r^2$$

Example: Find the centre and radius of the circle with equation $(x - 2)^2 + (y + 3)^2 = 16$.

Compare the equation... $(x - 2)^2 + (y + 3)^2 = 16$
...with the general form: $(x - a)^2 + (y - b)^2 = r^2$

So $a = 2$, $b = -3$ and $r = 4$.

So the centre (a, b) is **(2, –3)** and the radius r is **4**.

Example: Write down the equation of the circle with centre (–4, 2) and radius 6.

The question says, 'Write down...', so you know you don't need to do any working.

The centre of the circle is (–4, 2), so $a = -4$ and $b = 2$.
The radius is 6, so $r = 6$.

Using the general equation $(x - a)^2 + (y - b)^2 = r^2$
you can write: $\mathbf{(x + 4)^2 + (y - 2)^2 = 36}$

Complete the Square to get into the *Familiar Form*

Not all circle equations look like $(x - a)^2 + (y - b)^2 = r^2$. If they don't, it can be a bit of a pain, because you can't immediately tell what the **radius** is or where the **centre** is. But all it takes is a bit of **rearranging**.

Example: Write the equation $x^2 + y^2 - 6x + 4y + 4 = 0$ in the form $(x - a)^2 + (y - b)^2 = r^2$.

Complete the square on the x and y terms.

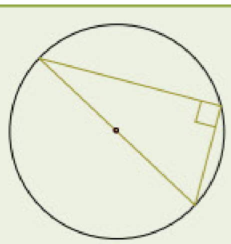
$$x^2 + y^2 - 6x + 4y + 4 = 0$$
$$x^2 - 6x + y^2 + 4y + 4 = 0$$ ← Collect the x and y terms together...
$$(x - 3)^2 - 9 + (y + 2)^2 - 4 + 4 = 0$$ ← ...then find squares that give the terms you need, and add constants to balance things up.
$$\mathbf{(x - 3)^2 + (y + 2)^2 = 9}$$

Have a look at page 16 for more on completing the square.

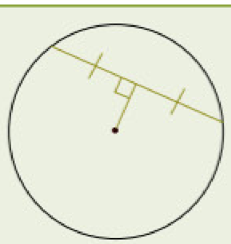
This is the recognisable form, so the centre is **(3, –2)** and the radius is $\sqrt{9}$ **= 3**.

Don't forget the *Properties of Circles*

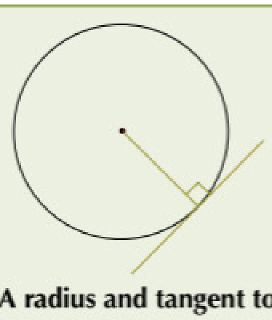
You will have seen the circle properties at GCSE. You'll sometimes need to dredge them up from the darkest depths of your memory for these circle questions. Here's a reminder of the ones you need to know for this course.



The angle in a semicircle is a right angle.



The perpendicular from the centre to a chord bisects the chord.



A radius and tangent to the same point will meet at right angles.

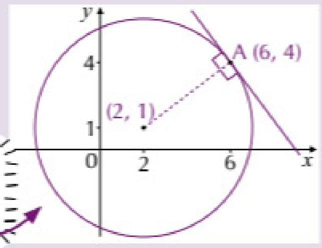
Bob thinks the Magic Circle rules. Bunnykin isn't so sure.

Geometry of Lines and Circles

*Use the **Gradient Rule** for **Perpendicular Lines***

Remember that the tangent at a given point will be perpendicular to the radius at that same point.

Example: Point A (6, 4) lies on a circle with the equation $(x-2)^2+(y-1)^2=25$. Find the equation of the tangent to the circle at A.



The equation of the circle tells you the centre is (2, 1).
The tangent you're interested in is at right angles to the radius at (6, 4).

The gradient of the **radius** at (6, 4) $= \frac{4-1}{6-2} = \frac{3}{4}$,

so the gradient of the **tangent** at (6, 4) $= \frac{-1}{\frac{3}{4}} = -\frac{4}{3}$.

It often helps with questions like this to use what you know at the start to sketch the graph.

So using $y - y_1 = m(x - x_1)$, the tangent at (6, 4) is: $y - 4 = -\frac{4}{3}(x-6) \Rightarrow 3y - 12 = -4x + 24 \Rightarrow$ $\mathbf{3y + 4x - 36 = 0}$

Practice Questions

Q1 Find the equations of the lines that pass through the points: a) (2, –1) and (–4, –19), b) $\left(0, -\frac{1}{3}\right)$ and $\left(5, \frac{2}{3}\right)$.
Write each answer in the forms:
(i) $y - y_1 = m(x - x_1)$, (ii) $y = mx + c$, (iii) $ax + by + c = 0$, where a, b and c are integers.

Q2 The line l has equation $y = \frac{3}{2}x - \frac{2}{3}$. Find the equation of the line parallel to l, going through the point (4, 2).

Q3 The line k passes through the point (6, 1) and is perpendicular to $2x - y - 7 = 0$. What is the equation of k?

Q4 The coordinates of points R and S are (–8, 15) and (10, 3) respectively. Find, in the form $y = mx + c$, the equation of the line perpendicular to RS, passing through the midpoint of RS.

Q5 Write the equation of the circle with centre (3, –1) and radius 7.

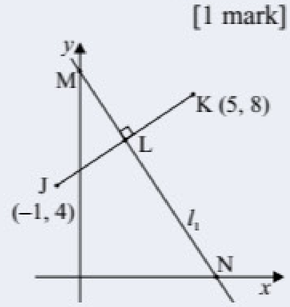
Q6 Give the radius and the coordinates of the centre of the circles with the following equations:
a) $x^2 + y^2 = 9$ b) $(x-2)^2 + (y+4)^2 = 4$ c) $x(x+6) = y(8-y)$

Exam Questions

Q1 The line segment PQ has equation $4x + 3y = 15$, where P has coordinates $(0, p)$ and Q has coordinates $(q, -3)$.
a) Find: (i) the gradient of PQ, (ii) the length of PQ. [3 marks]
b) The point R is the midpoint of PQ. Find the equation of the line which passes through the point R and is perpendicular to PQ, giving your answer in the form $y = mx + c$. [3 marks]

Q2 The line l passes through the point S (7, – 3) and has gradient – 2.
a) Find an equation of l, giving your answer in the form $y = mx + c$. [2 marks]
b) The point T has coordinates (5, 1). Show that T lies on l. [1 mark]

Q3 The points J and K have coordinates (–1, 4) and (5, 8) respectively. The line l_1 passes through the midpoint, L, of the line segment JK, and is perpendicular to JK, as shown.



a) Find an equation for l_1 in the form $ax + by + c = 0$, where a, b, and c are integers. [5 marks]
The line l_1 intersects the y-axis at the point M and the x-axis at the point N.
b) Find the coordinates of M. [2 marks]
c) Find the coordinates of N. [2 marks]

Q4 C is a circle with the equation: $x^2 + y^2 - 2x - 10y + 21 = 0$.
a) Find the centre and radius of C. [5 marks]
b) The line joining P (3, 6) and Q $(q, 4)$ is a diameter of C. Show that $q = -1$. [3 marks]
c) Find the equation of the tangent to C at Q, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers. [3 marks]

A Geometry of Lines and Circles, Book 1 — A Game of Maths...

Well, that sure was a lot of geometry to deal with. Just make sure you've got all of these formulas and properties committed to memory for the exam. After all, when you play the Game of Maths, you win or you get no marks.