Binomial Expansions

If you're feeling a bit stressed, just take a couple of minutes to relax before trying to get your head round this page — it's a bit of a stinker in places. Have a cup of tea and think about something else for a couple of minutes. Ready...

*Writing **Binomial Expansions** is all about **Spotting Patterns***

Doing binomial expansions just involves **multiplying out** brackets. It would get nasty when you raise the brackets to **higher powers** — but once again I've got a **cunning plan**...

$(1+x)^0 = 1$

$(1+x)^1 = 1 + x$

$(1+x)^2 = 1 + 2x + x^2$

$(1+x)^3 = 1 + 3x + 3x^2 + x^3$

$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$

Anything to the power of 0 is 1.

$(1+x)^3 = (1+x)(1+x)^2$
$= (1+x)(1+2x+x^2)$
$= 1 + 2x + x^2 + x + 2x^2 + x^3$
$= 1 + 3x + 3x^2 + x^3$

A Frenchman named Pascal spotted the pattern in the coefficients and wrote them down in a **triangle**.
So it was called '**Pascal's Triangle**' (imaginative, eh?).
The pattern's easy — each number is the **sum** of the two above it.
So, the next line will be: **1 5 10 10 5 1**
giving $\mathbf{(1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5}$.

				1				
			1		1			
		1		2		1		
	1		3		**3 +**		**1**	
1		4		6		**= 4**		1

*You **Don't** need to write out Pascal's Triangle for **Higher Powers***

There's a **formula** that gives you the numbers from the triangle. It looks **horrible** (take a glance at the next page — blegh...) but fortunately some kind soul has included it on your formula sheet. Make sure to say thank you.

Example: Expand $(1+x)^{20}$, giving the first four terms in ascending powers of x.

This includes the x^0 term (i.e. the constant term).

Here's the basic expansion for $(1+x)^n$. In this example $n = 20$.

$$(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1\times 2}x^2 + \boxed{\frac{n(n-1)(n-2)}{1\times 2\times 3}x^3} + \ldots + x^n$$

This binomial expansion stuff will be really important in some of the stats chapters — see p.161.

Here's a closer look at the term in the blue box:

Start here. The power of x is 3 and everything else here is based on 3.

$$\frac{n(n-1)(n-2)}{1\times 2\times 3}x^3$$

There are three things multiplied together on the top row. If n = 20, this would be 20 × 19 × 18.

There are three integers here multiplied together. 1 × 2 × 3 is written as 3! and called 3 factorial.

This means, if $n = 20$ and you were asked for '**the term in x^7**' you should write $\frac{20\times 19\times 18\times 17\times 16\times 15\times 14}{1\times 2\times 3\times 4\times 5\times 6\times 7}x^7$.

This can be **simplified** to $\frac{20!}{7!13!}x^7$.

$20 \times 19 \times 18 \times 17 \times 16 \times 15 \times 14 = \frac{20!}{13!}$ because it's the numbers from 20 to 1 multiplied together, divided by the numbers from 13 to 1 multiplied together.

Believe it or not, there's an even **shorter** form: $\frac{20!}{7!13!}$ is written as ${}^{20}C_7$ or $\binom{20}{7}$

Your calculator should have an nCr function for working this out.

$${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$$\binom{n}{0} = \binom{n}{n} = 1$$

So, to finish the example, $(1+x)^{20} = 1 + \frac{20}{1}x + \frac{20\times 19}{1\times 2}x^2 + \frac{20\times 19\times 18}{1\times 2\times 3}x^3 + \ldots$

$= 1 + \binom{20}{1}x + \binom{20}{2}x^2 + \binom{20}{3}x^3 + \ldots$

$= \mathbf{1 + 20x + 190x^2 + 1140x^3 + \ldots}$

This is the way you'll normally see it written, since you'd usually just get your calculator to do all the hard work multiplying and dividing for you.

Binomial Expansions

There's a General Formula for Expanding $(a + b)^n$

So far, you've seen the expansion for $(1 + x)^n$. The **general formula** works on anything that looks like $(a + b)^n$, as long as n is a **positive integer**. This is the one that's given on your formula sheet, and it looks like this:

$$(a + b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \ldots + {}^nC_r a^{n-r}b^r + \ldots + b^n \quad n \in \mathbb{N}$$

This bit just means that n has to be a 'natural number' — basically, a positive integer.

If you set a = 1 and b = x, then you'd get your original binomial expansion formula. Go on — give it a go if you don't believe me.

These are those $\binom{n}{r}$ fellas that you met at the bottom of the last page.

Example: a) Find the first four terms in the expansion of $(2 - 3x)^6$, in ascending powers of x.

Here, $a = 2$, $b = (-3x)$ and $n = 6$. Using the formula:

$$(2 - 3x)^6 = 2^6 + \binom{6}{1}2^5(-3x) + \binom{6}{2}2^4(-3x)^2 + \binom{6}{3}2^3(-3x)^3 + \ldots$$
$$= 64 + (6 \times 32 \times -3x) + (15 \times 16 \times 9x^2) + (20 \times 8 \times -27x^3) + \ldots$$
$$= \mathbf{64 - 576x + 2160x^2 - 4320x^3 + \ldots}$$

b) Use your answer to give an estimate for the value of 1.7^6.

$(2 - 3x)^6 = 1.7^6$ when $x = 0.1$. Substitute this into your expansion:

$$1.7^6 \approx 64 - 576(0.1) + 2160(0.1)^2 - 4320(0.1)^3$$
$$= 64 - 57.6 + 21.6 - 4.32$$
$$= \mathbf{23.68}$$

The actual value of 1.7^6 is 24.137... so this is a pretty good estimate.

Estimates like this one work best when x is small, so that x^n gets tiny as n gets bigger — that way, the terms that you miss out won't affect the answer as much.

Example: Find the coefficient of x^9 in the expansion of $(4 - 2x)^{11}$.

You could work out the whole expansion up to x^9, but it's much quicker to just use one bit of the formula: $(a + b)^n = \ldots + \binom{n}{r}a^{n-r}b^r + \ldots$

$$(4 - 2x)^{11} = \ldots + \binom{11}{9}4^{11-9}(-2x)^9 + \ldots = \ldots + (55 \times 16 \times -512x^9) + \ldots = \ldots - 450\,560x^9 + \ldots$$

So the coefficient of x^9 is **−450 560**.

Watch out — the **term** is "$-450\,560x^9$", but the **coefficient** is just "−450 560". Make sure you've checked what the question's asking for.

Practice Questions

Q1 Write down the sixth row of Pascal's triangle (hint: it starts with a '1').

Q2 Give the first four terms in the expansion of $(1 + x)^{12}$, in ascending powers of x.

Q3 What are the first four terms in the expansion of $(1 - 2x)^{16}$, in ascending powers of x?

Q4 Find the first four terms in the expansion of $(2 + 3x)^5$, in ascending powers of x.

Exam Questions

Q1 Find the first five terms in the binomial expansion of $(4 + 3x)^6$, in ascending powers of x. [5 marks]

Q2 The coefficient of the x^3 term in the binomial expansion of $(1 + px)^7$ is 280.
Find the value of p. [3 marks]

Pascal was great at maths but bad at music — he only played the triangle...

That nCr function is pretty bloomin' useful (remember that it could be called something else on your calculator) — it saves a lot of button pressing and errors. Just make sure you put your numbers in the right way round.