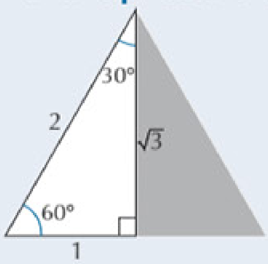
Trig Functions and Graphs

Questions on trigonometry quite often use the same common angles — so it pays to know the sin, cos and tan of them.

Draw Triangles to remember sin, cos and tan of the Important Angles

You should know the values of **sin**, **cos** and **tan** at 30°, 60° and 45°. But to help you remember, you can draw these two triangles. It may seem a complicated way to learn a few numbers, but it does make it easier. Honest. The idea is you draw the triangles below, putting in their angles and side lengths. Then you can use them to work out trig values like **sin 45°** or **cos 60°** more accurately than a calculator (which only gives a few decimal places).

Half an equilateral triangle with sides of length 2:



Get the height $\sqrt{3}$ by Pythagoras' Theorem:

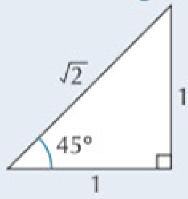
$1^2 + (\sqrt{3})^2 = 2^2$

Then you can use the triangle to work out sin, cos and tan of 30° and 60°.

Right-angled triangle with two sides of length 1:

The $\sqrt{2}$ just comes from Pythagoras.

This triangle gives you sin, cos and tan of 45°.



Remember: SOH CAH TOA...

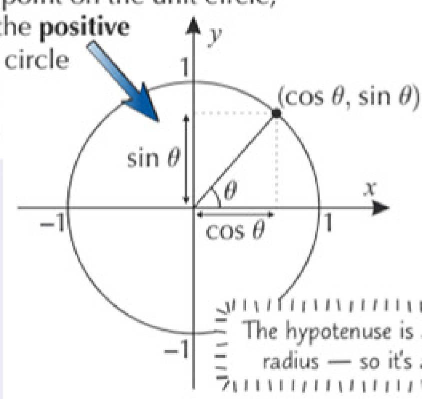
$\sin = \frac{\text{opp}}{\text{hyp}}$ $\cos = \frac{\text{adj}}{\text{hyp}}$ $\tan = \frac{\text{opp}}{\text{adj}}$

Trig Values from Triangles

$\sin 30° = \frac{1}{2}$	$\sin 60° = \frac{\sqrt{3}}{2}$	$\sin 45° = \frac{1}{\sqrt{2}}$
$\cos 30° = \frac{\sqrt{3}}{2}$	$\cos 60° = \frac{1}{2}$	$\cos 45° = \frac{1}{\sqrt{2}}$
$\tan 30° = \frac{1}{\sqrt{3}}$	$\tan 60° = \sqrt{3}$	$\tan 45° = 1$

*Find angles from the **Unit Circle***

The **unit circle** is a circle with **radius 1**, centred on the **origin**. For any point on the unit circle, the coordinates are **(cos θ, sin θ)**, where θ is the angle measured from the **positive** x-axis in an **anticlockwise** direction. The points on the **axes** of the unit circle give you the values of sin and cos of 0° and 90°. So at the point (1, 0): **cos 0° = 1**, **sin 0° = 0**. And at the point (0, 1): **cos 90° = 0**, **sin 90° = 1**.



The hypotenuse is always the radius — so it's always 1.

Example: The coordinates of a point on the unit circle, given to 3 s.f., are (0.788, 0.616). Find θ to the nearest degree.

The point is on the unit circle, so you know that the coordinates are (cos θ, sin θ). So cos θ = 0.788 and sin θ = 0.616.

You only need one of these to find the value of θ.

$\cos\theta = 0.788 \Rightarrow \theta = \cos^{-1}(0.788) =$ **38°** (to the nearest degree).

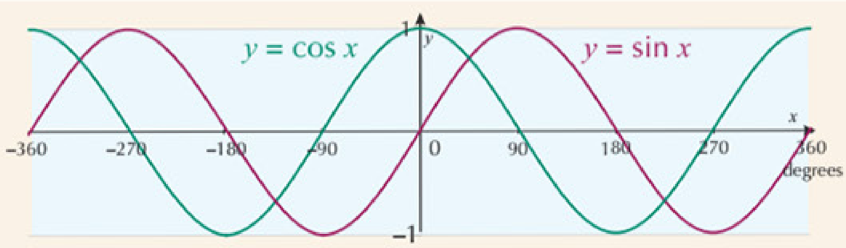
sin x** and **cos x** are always in the range **–1 to 1

sin x and **cos x** are similar — they just bob up and down between –1 and 1.

They bounce up and down from –1 to 1 — they can never have a value outside this range.

sin x and cos x are both **periodic** (repeat themselves) with **period 360°**

$\cos(x + 360°) = \cos x$ $\sin(x + 360°) = \sin x$



sin x goes through the **origin** — that means **sin 0 = 0**.

cos x crosses the y-axis at y **= 1** — that means **cos 0 = 1**.

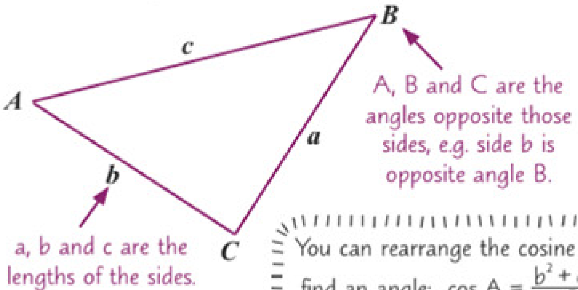
Symmetry in the **vertical** axis: $\cos(-x) = \cos x$ $\sin(-x) = -\sin x$

Trig Formulas and Identities

There are some more trig formulas you need to know for the exam.
So here they are — learn them or you're seriously stuffed. Worse than an aubergine.

*The **Sine Rule** and **Cosine Rule** work for **Any** triangle*

Remember, these three formulas work for **ANY** triangle, not just right-angled ones.



THE SINE RULE

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

This also works the other way up: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

THE COSINE RULE

$$a^2 = b^2 + c^2 - 2bc \cos A$$

You can rearrange the cosine rule to find an angle: $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

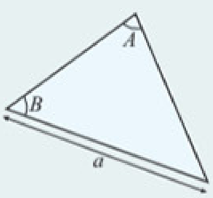
AREA OF ANY TRIANGLE

$$\text{Area} = \frac{1}{2}ab \sin C$$

***Sine Rule** or **Cosine Rule** — which one is it...*

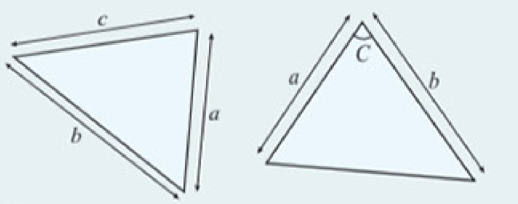
To decide which of these two rules you need to use, look at what you **already** know about the triangle.

Sine Rule



If you know **two angles** and a **side**.

Cosine Rule

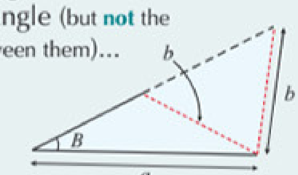


If you know **three sides**...

...or **two sides** and the **angle** between them.

Err — it doesn't work here...

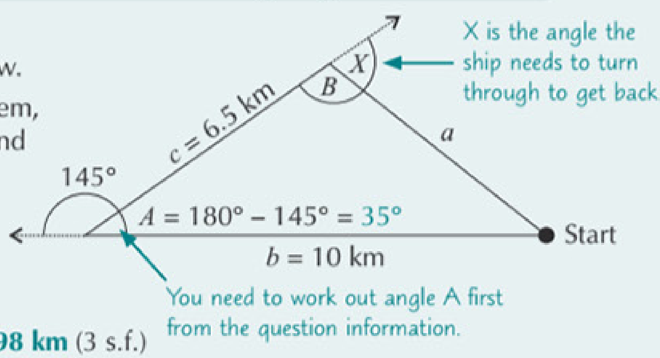
If you've got two sides and an angle (but **not** the angle between them)...



...there are **two possible triangles**.

Example: A ship sails due West for 10 km before turning clockwise through an angle of 145° and sailing in a straight line for another 6.5 km. Find the shortest distance back to its starting point, and the angle it would need to turn through to get there.

1) First draw a sketch of the problem, labelling all the lengths and angles you know.



2) You know 2 sides and the angle between them, so you're going to need the **cosine rule** to find side a, the distance back to the start:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$a^2 = 10^2 + 6.5^2 - 2(10)(6.5) \cos 35°$

$\Rightarrow a^2 = 142.25 - 130 \cos 35°$

$\Rightarrow a^2 = 35.7602... \Rightarrow a = 5.9799... =$ **5.98 km** (3 s.f.)

3) Now that you've got all the sides, you can use the **cosine rule** again to find angle B:

$\cos B = \frac{a^2 + c^2 - b^2}{2ac} \Rightarrow \cos B = \frac{35.7602... + 42.25 - 100}{2 \times 5.9799... \times 6.5}$

$\Rightarrow \cos B = \frac{-21.979...}{77.739...} = -0.2828...$

$\Rightarrow B = \cos^{-1} -0.2828... =$ **106.43...°**

4) So the angle, X, that the ship needs to turn through is $180° - 106.43...° =$ **73.6°** (1 d.p.)

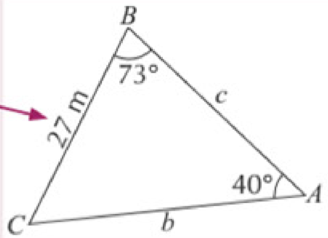
You could use the sine rule to find angle B, but watch out if you do — any value of $\sin \theta$ in the range $0 < \sin \theta < 1$ corresponds to two values of θ between 0° and 180°. Your calculator will give you the acute angle for B, but in this case you actually want the obtuse angle instead.

Trig Formulas and Identities

Label *the* ***Angles*** *and* ***Sides*** *carefully when* ***Sketching*** *triangles*

Example: In the triangle ABC, $A = 40°$, $a = 27$ m and $B = 73°$.
Find the missing angles and sides, and calculate the area of the triangle.

1) Draw a quick sketch first — don't worry if it's not deadly accurate, though.



Make sure you put side a opposite angle A.

2) You're given 2 angles and a side, so you need the **sine rule**.

The angles in a triangle add up to 180°.

First of all, get the other angle: $\angle C = 180° - 40° - 73° = $ **67°**

3) Then find the other sides, one at a time:

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{27}{\sin 40°} = \frac{b}{\sin 73°}$$

$$\Rightarrow b = \frac{\sin 73°}{\sin 40°} \times 27 = 40.169... = \mathbf{40.2\ m}\ (1\text{ d.p.})$$

$$\frac{c}{\sin C} = \frac{a}{\sin A} \Rightarrow \frac{c}{\sin 67°} = \frac{27}{\sin 40°}$$

$$\Rightarrow c = \frac{\sin 67°}{\sin 40°} \times 27 = 38.665... = \mathbf{38.7\ m}\ (1\text{ d.p.})$$

4) Now just use the formula to find its area:

$$\text{Area } \Delta ABC = \frac{1}{2}ab \sin C$$
$$= \frac{1}{2} \times 27 \times 40.169... \times \sin 67°$$
$$= \mathbf{499.2\ m^2}\ (1\text{ d.p.})$$

Use a more accurate value for b here, rather than the rounded value 40.2.

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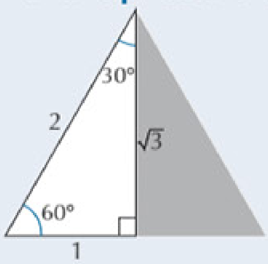

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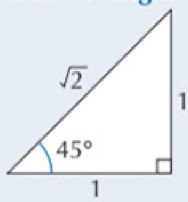

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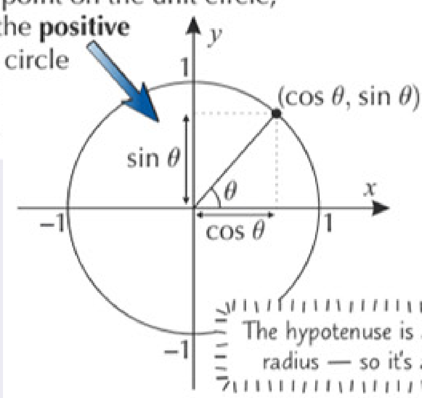

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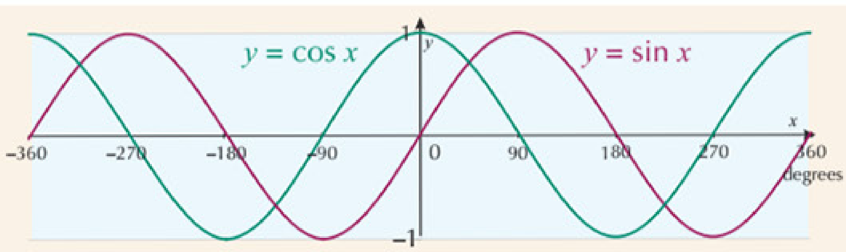

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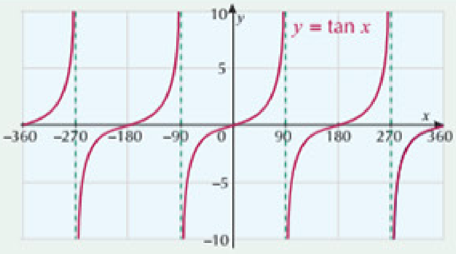
Symmetry in the **vertical** axis: $\cos(-x) = \cos x$ $\sin(-x) = -\sin x$

Trig Functions and Graphs

tan x can be *Any Value* at all

tan x is **different** from sin x or cos x. It doesn't go up and down between –1 and 1 — it goes **between –∞ and +∞**.

tan x is also periodic — but with **period 180°**



tan x goes from –∞ to +∞ **every 180°**

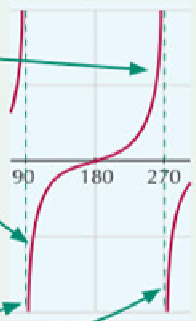
$\tan(x + 180°) = \tan x$

tan x is **undefined** at ±90°, ±270°, ±450°,...

As you approach one of these undefined points from the left, tan x just shoots up to **infinity**.

As you approach from the right, it drops to **minus infinity**.

The graph never ever touches these lines. But it does get infinitely close, if you see what I mean... These are called **asymptotes**.



The easiest way to sketch sin, cos or tan graphs is to plot the **important points** which happen **every 90°** (i.e. **–180°**, **–90°**, **0°**, **90°**, **180°**, **270°**, **360°**...) and then just join the dots up.

There are 3 basic types of *Transformed Trig Graph*

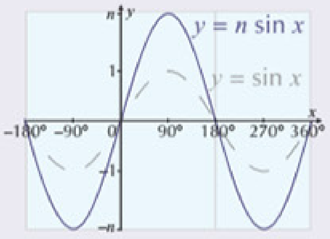
Transformed trigonometric graphs act just like the standard graph transformations on page 25.

$y = n \sin x$ — Vertical Stretch

If $n > 1$, the graph of $y = \sin x$ is **stretched vertically** by a factor of n.

If $0 < n < 1$, the graph is **squashed**.

And if $n < 0$, the graph is also **reflected** in the **x-axis**.

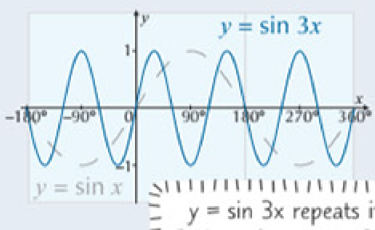


$y = \sin nx$ — Horizontal Stretch

If $n > 1$, the graph of $y = \sin x$ is **squashed horizontally** by a factor of n.

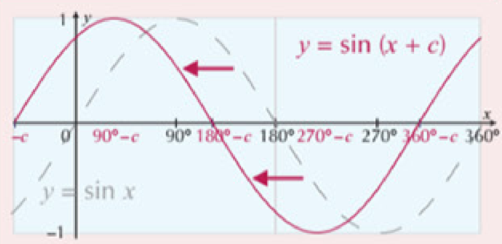
If $0 < n < 1$, the graph is **stretched**.

And if $n < 0$, the graph is also **reflected** in the **y-axis**.



y = sin 3x repeats itself 3 times from 0 to 360°.

$y = \sin(x + c)$ — Horizontal Translation



For $c > 0$, $\sin(x + c)$ is just $\sin x$ **translated c to the left**.

Similarly, $\sin(x - c)$ is just $\sin x$ **translated c to the right**.

Practice Questions

Q1 Write down the exact value of: a) cos 30°, b) sin 45°, c) tan 60°, d) sin 30°.

Q2 Sketch: a) $y = \frac{1}{2}\cos x$ for $0° \le x \le 360°$, b) $y = \sin(x + 30°)$ for $0 \le x \le 360$, c) $y = \tan 3x$ for $0° \le x \le 180°$.

Exam Questions

Q1 The coordinates of a point on the unit circle are (0.914, –0.407), to 3 s.f. Find the angle measured in an anticlockwise direction, from the positive x-axis to the radius from the origin to (0.914, –0.407), to the nearest degree. [2 marks]

Q2 a) Sketch, for $0 \le x \le 360°$, the graph of $y = \cos(x + 60°)$. [2 marks]
b) Write down all the values of x, for $0 \le x \le 360°$, where $\cos(x + 60°) = 0$. [2 marks]

Q3 Sketch, for $0 \le x \le 180°$, the graph of $y = \sin 4x$. [2 marks]

Curling up on the sofa with 2 cos x — that's my idea of cosiness...

It's really really really really really important that you can draw and transform the trig graphs on these pages. Trust me.