Exponentials and Logs

Don't be put off by your parents or grandparents telling you that logs are hard. Logarithm (log for short) is just a fancy word for power, and once you know how to use them you can solve all sorts of equations.

You need to be able to *Switch* between *Different Notations*

Exponentials and **logs** can describe the same thing because they are **inverses** of each other.

$\log_a b = c$ means the same as $a^c = b$

That means that $\log_a a = 1$ and $\log_a 1 = 0$

The little number 'a' after 'log' is called the **base**. Logs can be to any base, but **base 10** is the most common — this is usually left out, i.e. '$\log_{10}$' is just 'log'.

Example: Index notation: $10^2 = 100$ log notation: $\log_{10} 100 = 2$

So the **logarithm** of 100 to the **base 10** is 2, because 10 raised to the **power** of 2 is 100.

Your calculator might have a '$\log_{\blacksquare}\square$' button and a 'log' button.

Examples:

Write down the values of the following:
a) $\log_2 8$ b) $\log_5 5$

a) 8 is 2 raised to the power of 3, so $2^3 = 8$ and $\log_2 8 = 3$

b) Anything to the power of 1 is itself, so $\log_5 5 = 1$

Write the following using log notation:
a) $5^3 = 125$ b) $3^0 = 1$

a) 3 is the power or **logarithm** that 5 (the **base**) is raised to to get 125, so $\log_5 125 = 3$

b) You'll need to remember this one: $\log_3 1 = 0$

The *Laws of Logarithms* are *Unbelievably Useful*

Whenever you have to deal with **logs**, you'll probably end up using the **laws** below. That means it's not a bad idea to **learn them** by heart right now.

Laws of Logarithms		
$\log_a x + \log_a y = \log_a (xy)$	$\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$	$\log_a x^k = k \log_a x$

So $\log_a \frac{1}{x} = -\log_a x$

Use the *Laws* to *Manipulate Logs*

Example: Write each expression in the form $\log_a n$, where n is a number.
a) $\log_a 5 + \log_a 4$ b) $2\log_a 6 - \log_a 9$

a) $\log_a x + \log_a y = \log_a (xy)$

$\log_a 5 + \log_a 4 = \log_a (5 \times 4)$
$= \log_a 20$

b) $\log_a x^k = k \log_a x$

$2\log_a 6 = \log_a 6^2 = \log_a 36$
$\log_a 36 - \log_a 9 = \log_a (36 \div 9)$
$= \log_a 4$

You can use *Logs* to *Solve Equations*

Example: Solve $2^{4x} = 3$ to 3 significant figures.

You want x on its own, so take logs of both sides (by writing 'log' in front of both sides): $\log 2^{4x} = \log 3$

Use $\log x^k = k \log x$: $4x \log 2 = \log 3$

Divide both sides by '4 log 2' to get x on its own: $x = \frac{\log 3}{4 \log 2}$

But $\frac{\log 3}{4 \log 2}$ is just a number you can find using a calculator: $x = 0.396$ **(to 3 s.f.)**

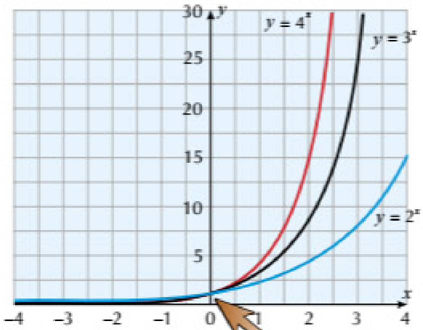
Alternatively, you could do:
$2^{4x} = 3 \Rightarrow \log_2 3 = 4x$
$\Rightarrow x = \frac{1}{4}\log_2 3 = 0.396$ (to 3 s.f.)

Exponentials and Logs

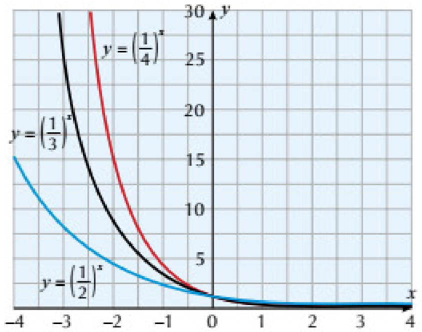
Graphs of a^x *never reach* ***Zero***

All the graphs of $y = a^x$ (**exponential graphs**) where $a > 1$ have the **same basic shape**. The graphs for $a = 2$, $a = 3$ and $a = 4$ are shown on the right.

- a is greater than 1 — so **y increases as x increases**.
- The **bigger** a is, the **quicker** the graphs increase.
- As x **decreases**, y **decreases** at a **smaller and smaller rate** — y will approach zero, but never actually get there.



All the graphs go through 1 at x = 0 because $a^0 = 1$ for any value of a.



The graphs on the left are for $y = a^x$ where $a < 1$ (they're for $a = \frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$).

- a is less than 1 — so **y decreases as x increases**.
- As x **increases**, y **decreases** at a **smaller and smaller rate** — again, y will approach zero, but never actually get there.

You can also use ***Exponentials*** *and* ***Logs*** *to* ***Solve Equations***

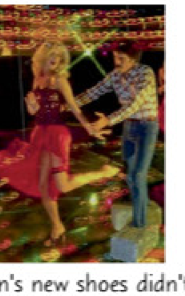
Example: Solve $7 \log_{10} x = 5$ to 3 significant figures.

You want x on its own, so begin by dividing both sides by 7: $\log_{10} x = \frac{5}{7}$

You now need to take exponentials of both sides by doing '10 to the power of both sides' (since the log is to base 10): $10^{\log_{10} x} = 10^{\frac{5}{7}}$

Logs and exponentials are inverse functions, so they cancel out: $x = 10^{\frac{5}{7}}$

But $10^{\frac{5}{7}}$ is just a number you can find using a calculator: **$x = 5.18$ (to 3 s.f.)**



John's new shoes didn't improve his logarithm.

Practice Questions

Q1 Write down the values of the following: a) $\log_3 27$ b) $\log_3 \left(\frac{1}{27}\right)$ c) $\log_3 18 - \log_3 2$

Q2 Simplify the following: a) $\log 3 + 2 \log 5$ b) $\frac{1}{2}\log 36 - \log 3$ c) $\log 2 - \frac{1}{4}\log 16$

Q3 Simplify $\log_b (x^2 - 1) - \log_b (x - 1)$

Q4 Solve these jokers to 4 significant figures: a) $10^x = 240$ b) $\log_{10} x = 5.3$ c) $10^{2x+1} = 1500$

Exam Questions

Q1 Solve the equation $\log_7 (y + 3) + \log_7 (2y + 1) = 1$, where $y > 0$. [4 marks]

Q2 a) Solve the equation $3^x = 5$, giving your answer to 2 decimal places. [3 marks]

b) Hence, or otherwise, solve the equation $3^{2x} - 14(3^x) = -45$ [4 marks]

It's sometimes hard to see the wood for the trees — especially with logs...

Tricky... I think of $\log_a b$ as 'the power I have to raise a to if I want to end up with b' — that's all it is. And the log laws make a bit more sense if you think of 'log' as meaning 'power'. For example, you know that $2^a \times 2^b = 2^{a+b}$ — this just says that if you multiply the two numbers, you add the powers. Well, the first law of logs is saying the same thing. Any road, even if you don't quite understand why they work, make sure you know the log laws like the back of your hand.

Using Exponentials and Logs

Now that you're familiar with the log laws, it's time to reveal their true power. Okay, maybe that's a slight exaggeration, but they are pretty useful in a variety of situations. Read on to find out more.

Use the Calculator Log Button whenever you can

Example: Use logarithms to solve the following for x, giving your answers to 4 s.f.
a) $10^{3x} = 4000$ b) $7^x = 55$ c) $\log_2 x = 5$

There's an unknown in the power, so take logs of both sides (if your calculator has a 'log$_{10}$' button, base 10 is usually a good idea): a) $\log 10^{3x} = \log 4000$

You can choose any base, but use the same one for both sides.

Use one of the log laws: $\log x^k = k \log x$ $3x \log 10 = \log 4000$

Since $\log_{10} 10 = 1$, solve the equation to find x: $3x = \log 4000$, so $x =$ **1.201 to 4 s.f.**

Again, take logs of both sides, and use the log rules: b) $x \log_{10} 7 = \log_{10} 55$, so $x = \frac{\log_{10} 55}{\log_{10} 7} =$ **2.059 to 4 s.f.**

To get rid of a log, you 'take exponentials', meaning you do '2 (the base) to the power of each side'. c) $2^{\log_2 x} = 2^5$, so $x = 2^5 =$ **32**

Or using base 7 for b): $x = \log_7 55 = 2.059$ (4 s.f.)

You might have to Combine the Laws of Logs to Solve equations

If the examiners are feeling particularly mean, they might make you use **more than one** law to solve an equation.

Example: Solve the equation $\log_3(2 - 3x) - 2\log_3 x = 2$.

First, combine the log terms into one term (you can do this because they both have the same base): $\log_3 \frac{2-3x}{x^2} = 2$

Remember that $a \log x = \log x^a$.

Then take exponentials of both sides: $3^{\log_3 \frac{2-3x}{x^2}} = 3^2 \Rightarrow \frac{2-3x}{x^2} = 9$

Finally, rearrange the equation and solve for x: $2 - 3x = 9x^2 \Rightarrow 0 = 9x^2 + 3x - 2$
$\Rightarrow 0 = (3x - 1)(3x + 2) \Rightarrow x = \frac{1}{3}$

Ignore the negative solution because you can't take logs of a negative number.

Exponential Growth and Decay applies to Real-life problems

Logs can even be used to model **real-life** situations — see pages 82-83 to see more of this.

Example: The radioactivity of a substance decays by 20 percent over a year. The initial level of radioactivity is 400. Find the time taken for the radioactivity to fall to 200 (the half-life).

$R = 400 \times 0.8^T$ where R is the **level of radioactivity** at time T years.
We need $R = 200$, so solve $200 = 400 \times 0.8^T$

The 0.8 comes from 100% – 20% decay.

$0.8^T = \frac{200}{400} = 0.5 \Rightarrow T \log 0.8 = \log 0.5 \Rightarrow T = \frac{\log 0.5}{\log 0.8} =$ **3.106 years** (4 s.f.)

Exponential **models** often have a **time restriction** — for larger times the numbers get too big or small.

Exponential equations can be Reduced to Linear Form

Equations like $y = ax^n$ and $y = ab^x$ can be a bit awkward to use. Fortunately, using the **laws of logs**, they can be rewritten to look like a form you've seen before — good old $y = mx + c$. Just take **logs** of both sides and rearrange:

$$y = ax^n \Rightarrow \log y = n \log x + \log a$$

$$y = ab^x \Rightarrow \log y = x \log b + \log a$$

The equations look pretty horrendous now, I'll admit. But look at them carefully — they're just a nasty-looking version of $y = mx + c$.

Once the equations are in this form you can draw their **straight line graphs** — you just need to **label** the axes **log x** (top) or **x** (bottom) against **log y**. Now your graph is **easier** to work with than the exponential graph.

e^x and ln x

Of all the exponential functions in the world (and there are infinite exponential functions), only one can be called the exponential function. The most powerful, most incredible — e^x. Wait, what do you mean, "anticlimactic"?

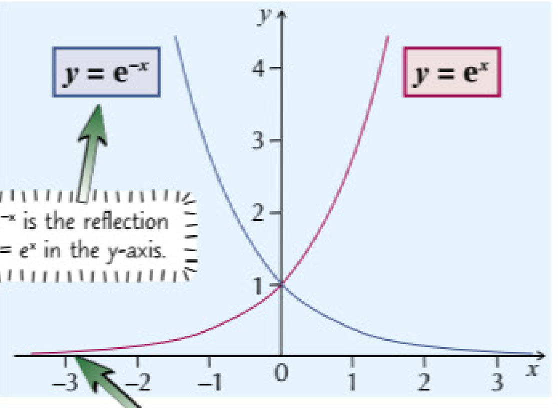
*The **Gradient** of the **Exponential Function** $y = e^x$ is e^x*

There is a value of 'a' for which the **gradient** of $y = a^x$ is **exactly the same as** a^x. That value is known as **e**, an **irrational number** around **2.7183** (it's stored in your calculator, just like π). Because e is just a number, the graph of $y = e^x$ has all the properties of $y = a^x$...

1) $y = e^x$ crosses the y-axis at **(0, 1)**.
2) As $x \to \infty$, $e^x \to \infty$ and as $x \to -\infty$, $e^x \to 0$.
3) $y = e^x$ **does not exist** for $y \le 0$ (i.e. e^x **can't** be **zero** or **negative**).

The fact that the **gradient** of e^x is e^x (i.e. e^x is its own gradient) is used lots in the differentiation section — see page 96.

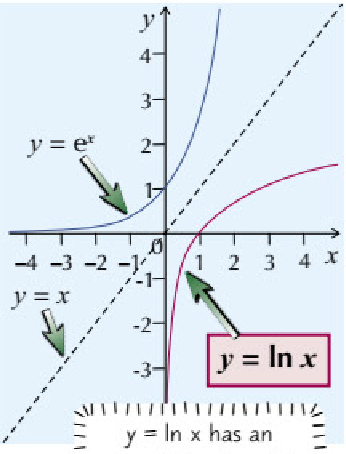
$y = e^{ax+b} + c$ is a transformation (see p.31) of $y = e^x$. The value of **a** stretches the graph horizontally, **b** shifts it horizontally and **c** shifts it vertically.



$y = e^{-x}$ is the reflection of $y = e^x$ in the y-axis.

e^x gets infinitely close to 0 but never reaches it — $y = 0$ is called an asymptote (see p.30) of the graph.

*ln x is the **Inverse Function** of e^x*



$y = \ln x$ has an asymptote at $x = 0$.

ln x (also known as $\log_e x$, or '**natural log**'*) is the **inverse function** of e^x (see p.34):

1) $y = \ln x$ is the **reflection** of $y = e^x$ in the line $y = x$.
2) It crosses the x-axis at **(1, 0)** (so **ln 1 = 0**).
3) As $x \to \infty$, $\ln x \to \infty$ (but slowly), and as $x \to 0$, $\ln x \to -\infty$.
4) $\ln x$ **does not exist** for $x \le 0$ (i.e. x **can't be zero or negative**).

Because ln x is a logarithmic function and the inverse of e^x, we get these juicy **formulas** and **log laws**...

$$e^{\ln x} = x$$
$$\ln(e^x) = x$$

i.e. doing one function then the other to x takes you back to x.

These formulas are **extremely useful** for dealing with **equations** containing 'e^x's or '$\ln x$'s.

'Log laws' for ln x

$$\ln x + \ln y = \ln xy$$
$$\ln x - \ln y = \ln\left(\frac{x}{y}\right)$$
$$\ln x^k = k \ln x$$

These are the same old log laws from p.76, applied to ln x.

*Use **Inverses** and **Log Laws** to **Solve Equations***

Just as with other logs and exponentials (p.76), you can use e^x and ln x to cancel each other out.

All logs and no play makes Jill a dull girl.

Example: If $e^{2x} = 9$, find the exact value of x.

Take ln of both sides: $\ln e^{2x} = \ln 9$

Using $\ln x^k = k\ln x$: $2x \ln e = \ln 9 \Rightarrow 2x = \ln 9$

$\Rightarrow x = \frac{1}{2}\ln 9 = \ln 9^{\frac{1}{2}} = \mathbf{\ln 3}$

The 'exact value of x' means 'leave x in terms of e and/or ln.'

Example: If $\ln(x - 5) = 3$, find the exact value of x.

Take exponentials of both sides: $e^{\ln(x-5)} = e^3$

$x - 5 = e^3$

$\Rightarrow x = 5 + e^3$

*Certified organic

e^x and ln x

You might have to solve a ***Quadratic Equation*** with **e** or ***ln***

Example: a) Solve the equation $2 \ln x - \ln 2x = 6$, giving your answer as an exact value of x.

Use the log laws to simplify: $2 \ln x - \ln 2x = 6$
$\Rightarrow \ln x^2 - \ln 2x = 6 \Rightarrow \ln (x^2 \div 2x) = 6 \Rightarrow \ln\left(\frac{x}{2}\right) = 6$

Now apply the inverse function e^x to both sides — this will remove the $\ln\left(\frac{x}{2}\right)$: $e^{\ln\left(\frac{x}{2}\right)} = e^6 \Rightarrow \frac{x}{2} = e^6 \Rightarrow x = \mathbf{2e^6}$

Using $e^{\ln x} = x$ from the last page.

b) Find the exact solutions of the equation $e^x + 5e^{-x} = 6$.

When you're asked for more than one solution think quadratics. $e^x + 5e^{-x} = 6$

Multiply each part of the equation by e^x to get rid of that e^{-x}: $\Rightarrow e^{2x} + 5 = 6e^x \Rightarrow e^{2x} - 6e^x + 5 = 0$

Basic power laws — $(e^x)^2 = e^{2x}$ and $e^{-x} \times e^x = e^0 = 1$.

It starts to look a bit nicer if you substitute y for e^x: $y^2 - 6y + 5 = 0$

Factorise to find solutions: $(y - 1)(y - 5) = 0 \Rightarrow y = 1$ and $y = 5$

Put e^x back in: $e^x = 1$ and $e^x = 5$

Using $\ln e^x = x$.

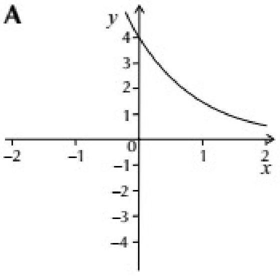
Take 'ln' of both sides to solve: $\ln e^x = \ln 1 \Rightarrow x = \ln 1 = \mathbf{0}$ and $\ln e^x = \ln 5 \Rightarrow \boldsymbol{x = \ln 5}$

Practice Questions

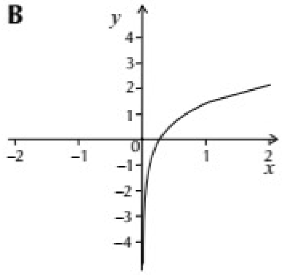
Q1 Four graphs, A, B, C and D, are shown below. Match the graphs to each of the following equations:

a) $y = 4e^x$ b) $y = 4e^{-x}$ c) $y = 4 \ln x$ d) $y = \ln 4x$

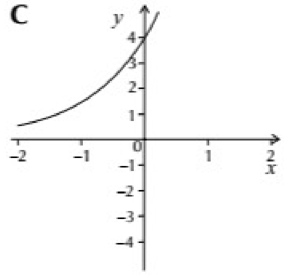
A



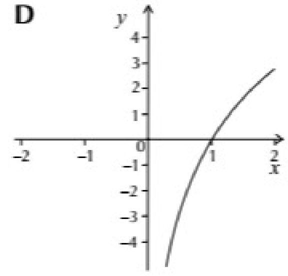
B



C



D



Q2 Find the value of x, to 4 decimal places, when:

a) $e^{2x} = 6$ b) $\ln (x + 3) = 0.75$ c) $3e^{-4x+1} = 5$ d) $\ln x + \ln 5 = \ln 4$

Q3 Solve the following equations, giving your solutions as exact values:

a) $\ln (2x - 7) + \ln 4 = -3$ b) $2e^{2x} + e^x = 3$

Exam Questions

Q1 a) Given that $6e^x = 3$, find the exact value of x. [2 marks]

b) Find the exact solutions to the equation $e^{2x} - 8e^x + 7 = 0$. [4 marks]

c) Given that $4 \ln x = 3$, find the exact value of x. [2 marks]

d) Solve the equation $\ln x + \frac{24}{\ln x} = 10$, giving your answers as exact values of x. [4 marks]

Q2 Solve the following equations, giving your answers as exact values of x.

a) $2e^x + 18e^{-x} = 20$ [4 marks]

b) $2 \ln x - \ln 3 = \ln 12$ [3 marks]

No problems — only solutions...

All the individual steps to solving these equations are easy — the hard bit is spotting what combination of things to try. A good thing to look for is hidden quadratics, so try and substitute for e^x or ln x to make things look a bit nicer. There's more about sketches on the next page, so don't worry too much if they're a bit confusing at the minute.