Exponential Growth *and* ***Decay*** *applies to* ***Real-life*** *problems*

Logs can even be used to model **real-life** situations — see pages 82-83 to see more of this.

Example: The radioactivity of a substance decays by 20 percent over a year. The initial level of radioactivity is 400. Find the time taken for the radioactivity to fall to 200 (the half-life).

$R = 400 \times 0.8^T$ where R is the **level of radioactivity** at time T years.

We need $R = 200$, so solve $200 = 400 \times 0.8^T$ ← The 0.8 comes from 100% – 20% decay.

$0.8^T = \frac{200}{400} = 0.5 \Rightarrow T \log 0.8 = \log 0.5 \Rightarrow T = \frac{\log 0.5}{\log 0.8} =$ **3.106 years** (4 s.f.)

Exponential **models** often have a **time restriction** — for larger times the numbers get too big or small.

Modelling with e^x and ln x

This page is all about models. Except they're modelling exponential growth and decay in real-world applications rather than the Chanel Autumn/Winter collection. Unless Chanel's lineup is growing exponentially, I suppose.

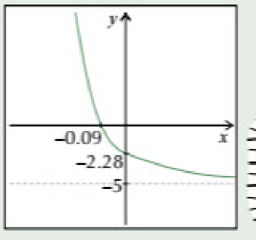
You can Sketch $y = e^{ax+b} + c$ and $y = \ln(ax + b)$

You should be familiar with the shape of the bog-standard exponential graphs, but most exponential functions will be **transformed** in some way. You need to know how the **key features** of the graph change depending on the function.

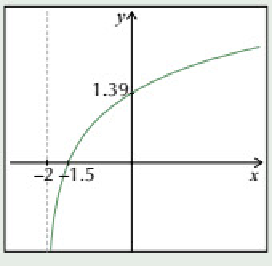
Example: Sketch the following functions, labelling any key points and giving the equations of any asymptotes.
a) $y = e^{-7x+1} - 5$ $(x \in \mathbb{R})$ and b) $y = \ln(2x + 4)$ $(x \in \mathbb{R})$.

a) $y = e^{-7x+1} - 5$

- '**Key points**' usually means where the graph crosses the axes, i.e. where x and y are 0:
  When $x = 0$, $y = e^1 - 5 = $ **−2.28** (3 s.f.)
  When $y = 0$, $e^{-7x+1} = 5 \Rightarrow -7x + 1 = \ln 5 \Rightarrow x = $ **−0.0871** (3 s.f.)
- Next see what happens as x tends to $\pm\infty$ to find any **asymptotes**:
  As $x \to \infty$, $e^{-7x+1} \to 0$, so $y \to -5$.
  As $x \to -\infty$, $e^{-7x+1} \to \infty$, so $y \to \infty$.
  y can't go below −5, so there's a horizontal asymptote at $y = -5$.
- Now use all the information to sketch out a graph.



$y > -5$ is the range of the function (see page 34).



b) $y = \ln(2x + 4)$

- First, the intercepts: When $x = 0$, $y = \ln 4 = $ **1.39** (3 s.f.)
  When $y = 0$, $2x + 4 = e^0 = 1 \Rightarrow x = $ **−1.5** (3 s.f.)
- As $x \to \infty$, $y \to \infty$ (gradually). As $x \to -\infty$, y decreases until $2x + 4 = 0$, at which point it can no longer exist (since $\ln x$ can only exist for $x > 0$). This gives an **asymptote** at $2x + 4 = 0$, i.e. $x = -2$.
- Sketch the graph using all of the information.

$x > -2$ is the domain (see p.34).

Use Exponential Functions to Model real-life Growth and Decay

In the **exam** you might be given a background story to an exponential equation.
They may then ask you to **find some values**, work out a **missing part** of the equation, or even **sketch a graph**.
There's nothing here you haven't seen before — you just need to know how to deal with all the **wordy** bits.

Example: The exponential growth of a colony of bacteria can be modelled by the equation $B = 60e^{0.03t}$, where B is the number of bacteria, and t is the time in hours from the point at which the colony is first monitored $(t \geq 0)$. Use the model to predict:

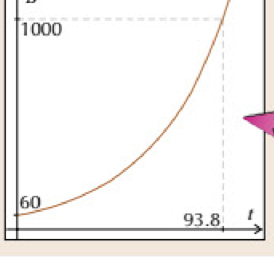
a) the number of bacteria after 4 hours,

You need to find B when $t = 4$, so put the numbers into the equation:

$B = 60 \times e^{(0.03 \times 4)}$
$= 60 \times 1.1274...$
$= 67.6498...$

So $B = $ **67 bacteria**

You shouldn't round up here — there are only 67 whole bacteria, not 68.



You might need to think about whether the model is suitable for large values of t — populations can't go on forever. A sketch of the function can help here.

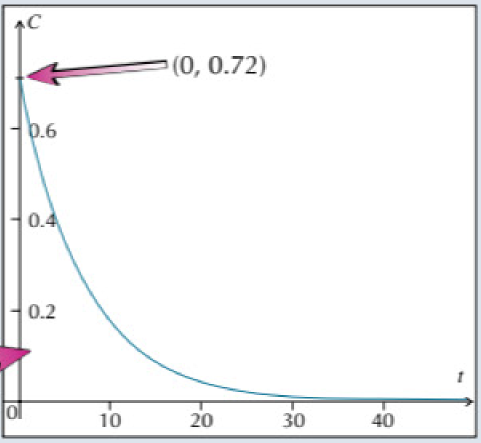
b) the time taken for the colony to grow to 1000.

- You need to find t when $B = 1000$, so put the numbers into the equation:
  $1000 = 60e^{0.03t}$
  $\Rightarrow e^{0.03t} = 1000 \div 60 = 16.6666...$
- Now take 'ln' of both sides as usual:
  $\ln e^{0.03t} = \ln(16.6666...)$
  $\Rightarrow 0.03t = 2.8134...$
  $\Rightarrow t = 2.8134... \div 0.03 = $ **93.8 hours (3 s.f.)**

Modelling with e^x and $\ln x$

Example: The concentration (C) of a drug in the bloodstream, t hours after taking an initial dose, decreases exponentially according to $C = Ae^{-kt}$, where k is a constant. If the initial concentration is 0.72, and this halves after 5 hours, find the values of A and k and sketch the graph of C against t.

- The 'initial concentration' is 0.72 when $t = 0$, so put this information into the equation to find A:
  $0.72 = A \times e^0 \Rightarrow 0.72 = A \times 1 \Rightarrow$ $\mathbf{A = 0.72}$
- The question also says that when $t = 5$ hours, C is half of 0.72. So using the value for A found above:
  $C = 0.72e^{-kt}$
  $0.72 \div 2 = 0.72 \times e^{(-k \times 5)}$
  $\Rightarrow 0.36 = 0.72 \times e^{-5k} \Rightarrow 0.36 = \frac{0.72}{e^{5k}} \Rightarrow e^{5k} = \frac{0.72}{0.36} = 2$
- Now take 'ln' of both sides to solve:
  $\ln e^{5k} = \ln 2 \Rightarrow 5k = \ln 2$
  $\Rightarrow k = \ln 2 \div 5 =$ **0.139 (3 s.f.)**
- So the equation is $C = 0.72e^{-0.139t}$.
  You still need to do a **sketch** though, so find the intercepts and asymptotes as you did on the last page:
  When $t = 0$, $C = 0.72$. As $t \to \infty$, $e^{-0.139t} \to 0$, so $C \to 0$.



The sketch should make sense for the situation in the question — here t can only be positive as it is the time after an event, so only sketch the graph for t ≥ 0.

Practice Questions

Q1 Sketch graphs of the following, labelling key points and asymptotes:

a) $y = 2 - e^{x+1}$ b) $y = 5e^{0.5x} + 5$ c) $y = \ln(2x) + 1$ d) $y = \ln(x + 5)$

Q2 The value of a motorbike (£V) varies with age (in t years from new) according to $V = 7500e^{-0.2t}$.

a) How much did it originally cost?

b) What will its value be after 10 years (to the nearest £)?

c) After how many years will the motorbike's value have fallen to £500? Give your answer to 1 d.p.

d) Sketch a graph showing how the value of the motorbike varies with age, labelling all key points.

Exam Questions

Q1 A breed of mink is introduced to a new habitat.
The number of mink, M, after t years in the habitat, is modelled by: $M = 74e^{0.6t}$ $(t \geq 0)$

a) State the number of mink that were introduced to the new habitat originally. [1 mark]

b) Predict the number of mink after 3 years in the habitat. [2 marks]

c) Predict the number of complete years it would take for the population of mink to exceed 10 000. [2 marks]

d) Sketch a graph to show how the mink population varies with time in the new habitat. [2 marks]

Q2 A radioactive substance decays exponentially so that its activity, A, can be modelled by $A = Be^{-kt}$, where t is the time in days, and $t \geq 0$. Some experimental data is shown in the table.

t	0	5	10
A	50	42	

a) State the value of B. [1 mark]

b) Find the value of k, to 3 significant figures. [2 marks]

c) Estimate the missing value from the table, to the nearest whole number. [2 marks]

d) The half-life of a substance is the time it takes for the activity to halve.
Find the half-life of this substance, in days. Give your answer to the nearest day. [3 marks]

Learn this and watch your knowledge grow exponentially...

For these wordy problems, the key is just to extract the relevant information and solve like you did on the previous pages. The more you practise, the more familiar they'll become — soon you'll be able to do them with your eyes shut.