Integrating $f(x) = x^n$

Integration is the 'opposite' of differentiation — and so if you can differentiate, you can be pretty confident you'll be able to integrate too. There's just one extra thing you have to remember — the constant of integration...

The Fundamental Theorem of Calculus

When you differentiate y, you get $\frac{dy}{dx}$.
And when you integrate $\frac{dy}{dx}$, you get y plus a **constant of integration**.

$y\ (+\ C)$ — Differentiate → $\frac{dy}{dx}$
$\frac{dy}{dx}$ — Integrate → $y\ (+\ C)$

*You need the constant because there's **More Than One** right answer*

When you **integrate** something, you're trying to find a function that differentiates to give what you started with. You add the **constant of integration** to allow for the fact that there's **more than one** possible function that does this...

This means the **integral** of 2x **with respect to x**. → $\int 2x\,dx =$ $x^2 - 207.253$, $x^2 - 1$, x^2, $x^2 + \pi$

If you differentiate any of these functions, you get the thing on the left — they're **all** possible answers.

So the answer to this integral is actually... $\int 2x\,dx = x^2 + C$

The 'C' just means '**any number**'. This is the **constant of integration**.

You only need to add a constant of integration to **indefinite integrals** like these ones.
Definite integrals are just integrals with **limits** (or little numbers) next to the integral sign (see p.110).

*Up the power by **One** — then **Divide** by it*

The formula below tells you how to integrate **any** power of x (except x^{-1}).

This is an indefinite integral — it doesn't have any limits (numbers) next to the integral sign.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

You can't do this to $\frac{1}{x} = x^{-1}$. When you increase the power by 1 (to get **zero**) you end up dividing by zero — and that's a big problem.

See p.114 if you just can't wait to find out how to integrate x^{-1}.

In a nutshell, this says: To integrate a power of x: (i) increase the power by one — then divide by it, and (ii) stick a constant on the end.

Examples: Use the integration formula...

1 For 'normal' powers

$$\int x^3\,dx = \frac{x^4}{4} + C$$

Increase the power to 4... ...and then divide by 4.

2 For negative powers

$$\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = \frac{x^{-2}}{-2} + C = -\frac{1}{2x^2} + C$$

Increase the power by 1 to −2... ...and then divide by −2.

3 For fractional powers

$$\int \sqrt[3]{x^4}\,dx = \int x^{\frac{4}{3}}\,dx = \frac{x^{\frac{7}{3}}}{(7/3)} + C = \frac{3\sqrt[3]{x^7}}{7} + C$$

Add 1 to the power... ...then divide by this new power.

4 And for complicated looking stuff...

$$\int \left(3x^2 - \frac{2}{\sqrt{x}} + \frac{7}{x^2}\right)dx = \int (3x^2 - 2x^{-\frac{1}{2}} + 7x^{-2})\,dx = \frac{3x^3}{3} - \frac{2x^{\frac{1}{2}}}{(1/2)} + \frac{7x^{-1}}{-1} + C = x^3 - 4\sqrt{x} - \frac{7}{x} + C$$

Do each of these bits separately.

CHECK YOUR ANSWERS:
You can check you've integrated properly by **differentiating** the **answer** — you should end up with the thing you started with.

Integrating $f(x) = x^n$

*You sometimes need to find the **Value** of the **Constant of Integration***

When they tell you something else about the curve in addition to its derivative, you can work out the value of that **constant of integration**. Usually the something is the **coordinates** of one of the points the curve goes through.

Example: The curve $y = f(x)$ goes through the point (2, 8) and $f'(x) = 6x(x - 1)$. Find $f(x)$.

You know the derivative $f'(x)$ and need to find the function $f(x)$ — so **integrate**.

$$f'(x) = 6x(x - 1) = 6x^2 - 6x$$

So integrating both sides gives...

$$f(x) = \int (6x^2 - 6x)\,dx$$

$$\Rightarrow f(x) = 6\int (x^2 - x)\,dx$$

6 is a constant factor of both terms, so you can take it outside the integral.

$$\Rightarrow f(x) = 6\left(\frac{x^3}{3} - \frac{x^2}{2} + C\right)$$

$$\Rightarrow \mathbf{f(x) = 2x^3 - 3x^2 + C}$$

You don't need to write 6C here, as C is just 'some unknown number'.

f'(x) is just another way of saying $\frac{dy}{dx}$. So when you integrate f'(x) you get f(x).

Check this is correct by **differentiating** it and making sure you get what you started with.

$$f(x) = 2x^3 - 3x^2 + C = 2x^3 - 3x^2 + C$$
$$f'(x) = 2(3x^2) - 3(2x^1) + 0$$
$$f'(x) = 6x^2 - 6x$$

A constant always differentiates to zero.

Remember: Even if you don't have any extra information about the curve — you still have to add a constant when you work out an integral without limits.

So this function's got the **correct derivative** — but you now need to **find C**.

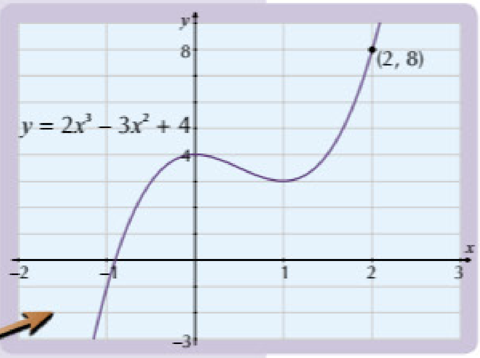
You do this using the fact that the curve goes through (2, 8).
Putting $x = 2$ and $f(x) = 8$ into the equation above gives:

$$8 = (2 \times 2^3) - (3 \times 2^2) + C$$
$$\Rightarrow 8 = 16 - 12 + C$$
$$\Rightarrow C = 4$$

So the answer you need is this one:

$$\mathbf{f(x) = 2x^3 - 3x^2 + 4}$$

It's a cubic equation — and the graph looks like this...



Practice Questions

Q1 Integrate: a) $\int 10x^4\,dx$, b) $\int (3x + 5x^2)\,dx$, c) $\int x^2(3x + 2)\,dx$

Q2 Find the equation of the curve that has derivative $\frac{dy}{dx} = 6x - 7$ and goes through the point (1, 0).

Q3 $f(x)$ passes through (1, 0) and $f'(x) = 3x^3 + 2$. Work out the equation of $f(x)$.

Exam Questions

Q1 a) Show that $(5 + 2\sqrt{x})^2$ can be written in the form $a + b\sqrt{x} + cx$, stating the values of the constants a, b and c. [3 marks]

b) Hence find $\int (5 + 2\sqrt{x})^2\,dx$. [3 marks]

Q2 Curve C has equation $y = f(x)$, $x \neq 0$, where the derivative is given by $f'(x) = x^3 - \frac{2}{x^2}$. The point P (1, 2) lies on C.

a) Find an equation for the tangent to C at the point P, giving your answer in the form $y = mx + c$, where m and c are integers. [4 marks]

b) Find $f(x)$. [4 marks]

Indefinite integrals — joy without limits...

This integration lark isn't so bad then — there are only a couple of things to remember and then you can do it no problem. But that constant of integration catches loads of people out — it's so easy to forget — and you'll definitely lose marks if you do forget it. You have been warned. Other than that, there's not much to it. Hurray.