Definite Integrals

Some integrals have limits (i.e. little numbers) next to the integral sign. You integrate them in exactly the same way — but you don't need a constant of integration. Much easier. And scrummier and yummier too.

Definite Integrals are like regular integrals, but with Limits

Definite integrals are ones that have little numbers on the top and bottom called **limits**. These are the values of x that you're 'integrating between'.

Finding a definite integral isn't really any harder than an indefinite one — there's just an **extra** stage you have to do. After you've integrated the function, you have to work out the value of this new function by **sticking in** the limits, and **subtracting** what the **bottom** limit gave you from what the **top** limit gave you.

Example: Evaluate $\int_1^3 (x^2 + 2)\,dx$.

This is the integral of $x^2 + 2$ "from 1 to 3".

Find the integral in the normal way, then use the limits:

$$\int_1^3 (x^2 + 2)\,dx = \left[\frac{x^3}{3} + 2x\right]_1^3$$

Put the integrated function in **square brackets** and rewrite the limits on the right-hand side.

$2 = 2x^0$ — so increase the power (to 1) and divide by 1 to get $2x$.

$$= \left(\frac{(3)^3}{3} + 2(3)\right) - \left(\frac{(1)^3}{3} + 2(1)\right)$$

Do '**top limit** minus **bottom limit**'.

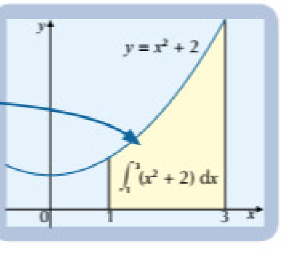
$$= 15 - \frac{7}{3} = \mathbf{\frac{38}{3}}$$

You don't need a constant of integration with a **definite** integral.

"I'm looking for a loan to start my business — Integrals, Ltd."

A Definite Integral finds the Area Under a Curve

1) Definite integrals give you the **area under the graph** of the function you're integrating. For instance, the integral in the example above gives this area:

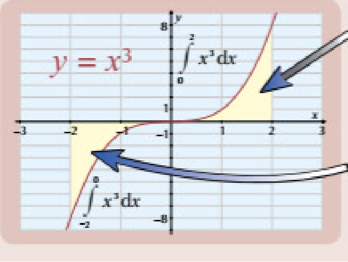


2) However, parts of the graph that are **below the x-axis** will give a **negative answer**, so you might need to split the integral up into bits. For example, if you wanted to find the area between the graph of $y = x^3$ and the x-axis between $x = -2$ and $x = 2$:

This is the right-hand side of the area you're finding...

$$\int_{-2}^{2} x^3\,dx$$

...and this is the left-hand side.

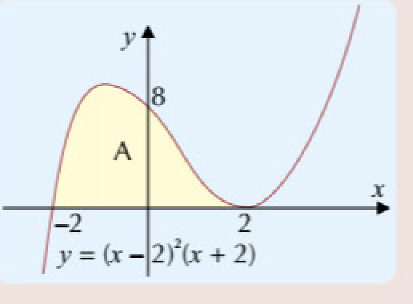


This bit is $\int_0^2 x^3\,dx = 4$. It's **positive** because the area is **above** the x-axis.

This bit is $\int_{-2}^0 x^3\,dx = -4$. It's **negative** because the area is **below** the x-axis.

The value of the integral $\int_{-2}^2 x^3\,dx$ is zero, because the area below the x-axis 'cancels out' the area above. To find the area, you need to work out the two parts separately and add them together. In this example, the area $= 4 + 4 = 8$.

Example: The curve $y = (x - 2)^2(x + 2)$ is shown on the diagram. Find the area bounded by the curve and the x-axis.



To find the area, you want to integrate $y = (x - 2)^2(x + 2)$ between -2 and 2.

Expand the brackets: $y = (x^2 - 4x + 4)(x + 2) = x^3 - 2x^2 - 4x + 8$

$$A = \int_{-2}^{2} y\,dx = \int_{-2}^{2} x^3 - 2x^2 - 4x + 8\,dx$$

Integrate each term separately.

$$= \left[\frac{x^4}{4} - \frac{2x^3}{3} - 2x^2 + 8x\right]_{-2}^{2}$$

$$= \left(\frac{2^4}{4} - \frac{2(2)^3}{3} - 2(2)^2 + 8(2)\right) - \left(\frac{(-2)^4}{4} - \frac{2(-2)^3}{3} - 2(-2)^2 + 8(-2)\right)$$

Stick the limits in.

$$= \frac{20}{3} - -\frac{44}{3}$$

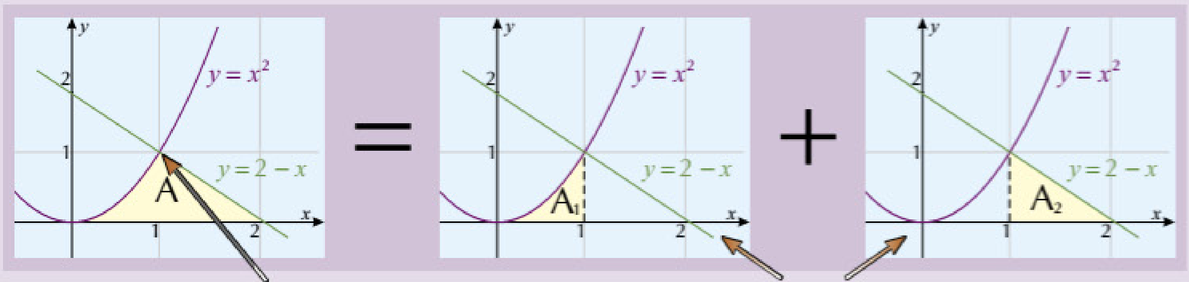
$$= \mathbf{\frac{64}{3}}$$

Definite Integrals

*Sometimes you have to **Add Two Integrals** together*

You could be asked to find "the area enclosed by [a couple of boring old curves] and the x-axis".
This might sound pretty hard — until you draw a picture and see what it's all about.
Then it's just a matter of choosing your limits wisely...

Example: Find the area enclosed by the curve $y = x^2$, the line $y = 2 - x$ and the x-axis.



Find out where the graphs meet by **solving** $x^2 = 2 - x$ — they meet at $x = 1$ (they also meet at $x = -2$, but this isn't in A).

You have to find area A — but you'll need to **split** it into two smaller pieces.

It's pretty clear from the picture that you'll have to find the area in two lumps, A_1 and A_2.

The first area you need to find is A_1:

$$A_1 = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \left(\frac{1}{3} - 0\right) = \mathbf{\frac{1}{3}}$$

The other area you need is A_2. A_2 is just a triangle, with base length $2 - 1 = 1$ and height $= 1$. So the area of the triangle is:

$$\frac{1}{2} \times b \times h = \frac{1}{2} \times 1 \times 1 = \mathbf{\frac{1}{2}}$$

And the area the question actually asks for is:

$$A = A_1 + A_2 = \frac{1}{3} + \frac{1}{2} = \mathbf{\frac{5}{6}}$$

Practice Questions

Q1 Evaluate the following definite integrals:

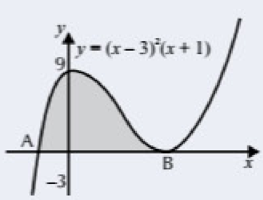
a) $\int_0^1 (4x^3 + 3x^2 + 2x + 1)\,dx$ b) $\int_1^2 \left(\frac{8}{x^5} + \frac{3}{\sqrt{x}}\right)dx$ c) $\int_1^6 \frac{3}{x^2}\,dx$.

Q2 a) Evaluate $\int_{-3}^3 (9 - x^2)\,dx$

b) Sketch the area represented by this integral.

Exam Questions

Q1 Find the exact value of $\int_1^4 (2x - 6x^2 + \sqrt{x})\,dx$. [5 marks]

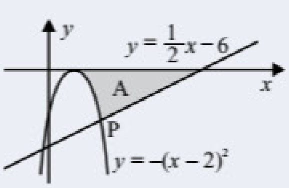


Q2 The diagram on the right shows a sketch of the curve C, $y = (x - 3)^2(x + 1)$. Calculate the shaded area between point A, where C intersects the x-axis, and point B, where C touches the x-axis. [7 marks]

Q3 The curve $y = -(x - 2)^2$ and line $y = \frac{1}{2}x - 6$ are shown on the diagram on the right.

a) Show that the curve and line intersect at the point P (4, –4). [3 marks]

b) Find the shaded area A, bounded by the curve $y = -(x - 2)^2$, the line $y = \frac{1}{2}x - 6$ and the x-axis. [6 marks]



My hobbies? Well, I'm really inte grating. Especially carrots.

It's still integration — but this time you're putting two numbers into an expression afterwards. So once you've got the hang of indefinite integration, this definite stuff should easily fall into place. Maths is like that. Though, I admit it's probably not as much fun as, say, a big banoffee cake. But Maths and cake together? Now we're talking...