Using Calculus for Kinematics

*Differentiate to find **Maximum** / **Minimum** values*

Stationary points are when the gradient is zero (see page 87) — you need to differentiate to find them.
To decide whether a stationary point is a **maximum** or a **minimum**, differentiate again.

Example: A particle sets off from the origin at $t = 0$ and moves in a straight line. At time t seconds, the velocity of the particle is v ms^{-1}, where $v = 9t - 2t^2$. Find the maximum velocity of the particle.

Differentiate v and put it equal to 0 to find any stationary points:

$v = 9t - 2t^2$

$\frac{dv}{dt} = 9 - 4t$

$\frac{dv}{dt} = 0$ when $t = \frac{9}{4} = $ **2.25 s**

If $\frac{d^2y}{dx^2} < 0$, then it's a maximum.
If $\frac{d^2y}{dx^2} > 0$, then it's a minimum.

Differentiate v again to decide whether this is a maximum or minimum:

$\frac{d^2v}{dt^2} = -4$, so $t = 2.25$ s is a **maximum**.

Substitute t = 2.25 s into the expression for v:

$v = 9(2.25) - 2(2.25^2) = 10.125$ ms^{-1}
So the maximum velocity is **10.125 ms^{-1}**.

Practice Questions

Q1 A particle moves along a straight line from the origin with velocity $v = 8t^2 - 2t$.
a) Find the acceleration of the particle at time t.
b) Find the displacement of the particle at time t.

Q2 A particle sets off from the origin at $t = 0$ s. Its displacement, in metres, at time t is $s = 6\sin\left(\frac{1}{3}t\right)$.
Find an expression for the acceleration of the particle at time t.

Q3 A particle is at rest at the origin at $t = 0$ s. It moves in a straight line with acceleration a ms^{-2}, where $a = e^{\frac{1}{4}t}$. Find the displacement of the particle at time $t = 8$ s.

Exam Questions

Q1 A model train sets off from a station at time $t = 0$ s. It travels in a straight line, then returns to the station.
At time t seconds, the distance, in metres, of the train from the station is $s = \frac{1}{100}(10t + 9t^2 - t^3)$, where $0 \le t \le 10$.
a) Sketch the graph of s against t and hence explain the restriction $0 \le t \le 10$ s. [3 marks]
b) Find the maximum distance of the train from the station. [5 marks]

Q2 A particle sets off from the origin O at $t = 0$ s and moves in a straight line.
At time t seconds, the velocity of the particle is v ms^{-1}, where

$$v = \begin{cases} 9t - 3t^2 & 0 \le t \le 2 \text{ s} \\ \frac{24}{t^2} & t > 2 \text{ s} \end{cases}$$

a) Find the maximum speed of the particle in the interval $0 \le t \le 2$ s. [4 marks]
b) Find the displacement of the particle from O at
(i) $t = 2$ s [3 marks]
(ii) $t = 6$ s [4 marks]

Calculus in kinematics — it's deriving me crazy...

This is one of those times when calculus is useful (told you so). The stuff you saw in Sections 7 and 8 can be applied to mechanics questions, so make sure you've really got calculus nailed. Then you've just got to remember that DISPLACEMENT differentiates to VELOCITY differentiates to ACCELERATION (and integrate to go the other way).

Using Calculus for Kinematics

The suvat equations you saw on page 186 are just grand when you've got constant acceleration. But when the acceleration varies with time, you need a few new tricks up your sleeve...

***Differentiate** to find **Velocity** and **Acceleration** from **Displacement**...*

If you've got a particle moving in a **straight line** with acceleration that **varies with time**, you need to use **calculus** to find equations to describe the motion — look back at Sections 7 and 8 for a reminder about calculus.

1) To find an equation for **velocity**, **differentiate** the equation for **displacement** with respect to time.
2) To find an equation for **acceleration**, **differentiate** the equation for **velocity** with respect to time or differentiate the equation for displacement with respect to time **twice**.

DISPLACEMENT (s) —Differentiate→ VELOCITY (v) —Differentiate→ ACCELERATION (a)

Displacement is sometimes written as x instead of s.

Example: A particle moves in a straight line. At time t seconds, the velocity of the particle is v ms^{-1}, where $v = 7t + 5t^2$. Find an expression for the acceleration of the particle at time t.

Velocity is given as a function of time, so differentiate to find the acceleration:

$$v = 7t + 5t^2$$

$$a = \frac{dv}{dt} = (7 + 10t)\text{ ms}^{-2}$$

$\frac{d}{dx}x^n = nx^{n-1}$

Tour de Kinematiques — Velo City, France.

*...and **Integrate** to find **Velocity** and **Displacement** from **Acceleration***

It's similar if you're trying to go "back the other way", except you **integrate** with respect to t rather than differentiate:

DISPLACEMENT (s) ←Integrate— VELOCITY (v) ←Integrate— ACCELERATION (a)

Example: A particle P sets off from O and moves in a straight line. At time t seconds, its velocity is v ms^{-1}, where $v = 12 - t^2$. At $t = 0$, displacement $s = 0$. Find the time taken for P to return to O.

Velocity is given as a function of t, so integrate to find the displacement:

$$s = \int v\,dt = 12t - \frac{t^3}{3} + C$$

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

Use the information given in the question to find the value of the constant:

$$0 = 12(0) - \frac{0^3}{3} + C \Rightarrow C = 0 \Rightarrow s = 12t - \frac{t^3}{3}$$

P is at O when s = 0, so solve the equation for t:

$$12t - \frac{t^3}{3} = 0 \Rightarrow t(36 - t^2) = 0 \Rightarrow t = 0, 6, \text{ or } -6$$

−6 can't be an answer, as you can't have a negative time.

Time taken for P to return to O is **6 seconds**.

Derive** the **suvat Equations** with **Calculus

You've derived the suvat equations with a v/t graph (p.188) — now it's time to use calculus.

Example: Use calculus to derive $v = u + at$ and $s = ut + \frac{1}{2}at^2$.

Acceleration is the rate of change of velocity v with time t: $a = \frac{dv}{dt} \Rightarrow v = \int a\,dt$

Carry out the integration (remember that **a** is a **constant**): $v = \int a\,dt = at + C$

Use the initial conditions v = u when t = 0 to find C: $u = a(0) + C$

$\Rightarrow C = u$

So $v = u + at$

Velocity is the rate of change of displacement s with time t: $v = \frac{ds}{dt} \Rightarrow s = \int v\,dt$

Substitute v = u + at and integrate: $s = \int (u + at)\,dt = ut + \frac{1}{2}at^2 + C$

Use the initial conditions s = 0 when t = 0 to find C: $0 = u(0) + \frac{1}{2}a(0)^2 + C \Rightarrow C = 0$

So $s = ut + \frac{1}{2}at^2$