Summary of key points

2
- If $y = \sin kx$, then $\frac{dy}{dx} = k\cos kx$
- If $y = \cos kx$, then $\frac{dy}{dx} = -k\sin kx$

3
- If $y = e^{kx}$, then $\frac{dy}{dx} = ke^{kx}$
- If $y = \ln x$, then $\frac{dy}{dx} = \frac{1}{x}$

4 If $y = a^{kx}$, where k is a real constant and $a > 0$, then $\frac{dy}{dx} = a^{kx}k\ln a$

5 The **chain rule** is: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

where y is a function of u and u is another function of x.

6 The chain rule enables you to differentiate a function of a function. In general,
- if $y = (f(x))^n$ then $\frac{dy}{dx} = n(f(x))^{n-1}f'(x)$
- if $y = f(g(x))$ then $\frac{dy}{dx} = f'(g(x))g'(x)$

7 $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

8 The **product rule**:
- If $y = uv$ then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$, where u and v are functions of x.
- If $f(x) = g(x)h(x)$ then $f'(x) = g(x)h'(x) + h(x)g'(x)$

9 The **quotient rule**:
- If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ where u and v are functions of x.
- If $f(x) = \frac{g(x)}{h(x)}$, then $f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{(h(x))^2}$

10
- If $y = \tan kx$, then $\frac{dy}{dx} = k\sec^2 kx$
- If $y = \operatorname{cosec} kx$, then $\frac{dy}{dx} = -k\operatorname{cosec} kx\cot kx$
- If $y = \sec kx$, then $\frac{dy}{dx} = k\sec kx\tan kx$
- If $y = \cot kx$, then $\frac{dy}{dx} = -k\operatorname{cosec}^2 kx$

15 You can use the chain rule to connect rates of change in situations involving more than two variables.

Differentiating standard functions

You need to **learn** these four derivatives for your A-level exam. They're not given in the formulae booklet and you need to be really confident using them.

Trigonometric functions

1. $\frac{d}{dx}(\sin x) = \cos x$
2. $\frac{d}{dx}(\cos x) = -\sin x$

These rules **only** work for angles measured in **radians**.

Exponential functions

3. $\frac{d}{dx}(e^x) = e^x$
4. $\frac{d}{dx}(\ln x) = \frac{1}{x}$

e^x is the only function that is the same when differentiated.

Worked example

Differentiate with respect to x

(a) $\sin(x^2 + 1)$ **(2 marks)**

$\frac{d}{dx}[\sin(x^2 + 1)] = 2x\cos(x^2 + 1)$

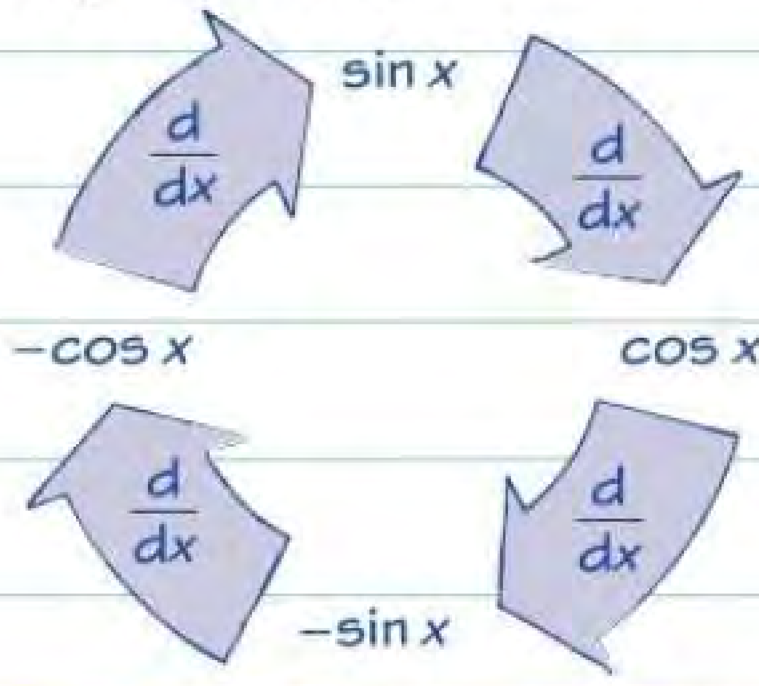
(b) $\sec x$ **(3 marks)**

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}(\cos x)^{-1}$$
$$= -(\cos x)^{-2} \times (-\sin x)$$
$$= \frac{\sin x}{\cos^2 x} = \sec x \tan x$$

You need the chain rule for both of these.

Round in circles

If you keep differentiating $\sin x$ you will end up back where you started.



Worked example

The function f is defined by

$$f : x \mapsto \ln(4 - x^2), \quad 0 \leqslant x < 2$$

Find the exact value of the gradient of the curve $y = f(x)$ at the point where it crosses the x-axis. **(5 marks)**

$$f'(x) = \frac{1}{4 - x^2} \times (-2x) = \frac{2x}{x^2 - 4}$$

When $f(x) = 0$, $\ln(4 - x^2) = 0$
$$4 - x^2 = 1$$
$$x = \sqrt{3} \text{ or } \cancel{x = -\sqrt{3}}$$

$$f'(\sqrt{3}) = \frac{2\sqrt{3}}{(\sqrt{3})^2 - 4} = -2\sqrt{3}$$

Worked example

The value of a boat, £V, after t years is modelled as

$$V = 12\,000e^{-\frac{1}{5}t}, \quad t > 0$$

(a) Find $\frac{dV}{dt}$ **(2 marks)**

$$\frac{dV}{dt} = 12\,000e^{-\frac{1}{5}t} \times \left(-\frac{1}{5}\right)$$
$$= -2400e^{-\frac{1}{5}t}$$

(b) Find the exact value of V when $\frac{dV}{dt} = -800$ **(3 marks)**

$$-800 = -2400e^{-\frac{1}{5}t}$$
$$e^{-\frac{1}{5}t} = \frac{1}{3}$$
$$-\frac{1}{5}t = \ln\left(\frac{1}{3}\right) = -\ln 3$$
$$t = 5\ln 3$$

Use $\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$, and $\sin^2 2y + \cos^2 2y = 1$

Now try this

1 Differentiate with respect to x

(a) $\sin 2x - \cos 4x$ Answer **(2 marks)**

(b) $x^2 - e^{4x-3}$ Answer **(3 marks)**

(c) $\ln(3x^2 + 1)$ Answer **(2 marks)**

2 Given that $x = \sin 2y$, show that

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1 - x^2}}$$ Answer **(4 marks)**

3 Use the chain rule to show that

$$\frac{d}{dx}(\sin^2 x) = \sin 2x$$ Answer **(3 marks)**

Had a look ☐ Nearly there ☐ Nailed it! ☐

Differentiating a^x

You can use $\frac{d}{dx}(e^x) = e^x$ to differentiate expressions involving e^x. You need to be more careful when differentiating powers of constants **other than** e.

$$\frac{d}{dx}(a^x) = a^x \ln a$$

You can learn this rule but you need to know how to **derive** it as well. Follow the steps in the box on the right to use e^x to differentiate a^x.

Two methods

1 You can use the laws of logs to differentiate a^x.

$y = a^x = e^{x \ln a}$

$\frac{dy}{dx} = \ln a \, e^{x \ln a} = a^x \ln a$

2 You can differentiate a^x implicitly.

$y = a^x$

$\ln y = \ln a^x = x \ln a$

$\frac{1}{y}\frac{dy}{dx} = \ln a$

$\frac{dy}{dx} = y \ln a = a^x \ln a$

Worked example

Given that $y = 3^x$, show that $\frac{dy}{dx} = 3^x \ln 3$ **(2 marks)**

$y = e^{x \ln 3}$

$\frac{dy}{dx} = \ln 3 \, e^{x \ln 3}$

$= 3^x \ln 3$

Make sure you're confident converting between a^x and $e^{x \ln a}$:

$a^x = e^{\ln a^x} = e^{x \ln a}$

Worked example

A curve has equation $2^x + y^2 = 2xy$

Find the exact value of $\frac{dy}{dx}$ at the point on C with coordinates (3, 2). **(7 marks)**

$2^x \ln 2 + 2y\frac{dy}{dx} = 2y + 2x\frac{dy}{dx}$

$2y\frac{dy}{dx} - 2x\frac{dy}{dx} = 2y - 2^x \ln 2$

$\frac{dy}{dx}(2y - 2x) = 2y - 2^x \ln 2$

$\frac{dy}{dx} = \frac{2y - 2^x \ln 2}{2y - 2x}$

At the point (3, 2), $\frac{dy}{dx} = \frac{2(2) - 2^3 \ln 2}{2(2) - 2(3)}$

$= -2 + 4 \ln 2$

You might have to differentiate a^x as part of an implicit differentiation. As long as you're not asked to show it, you can use the rule: $\frac{d}{dx}(a^x) = a^x \ln a$

Be really careful when you're substituting with implicit differentiation. Make sure you substitute the x- and y-values in the correct places in your expression for $\frac{dy}{dx}$.

There's more on implicit differentiation on page 94.

Now try this

1 Find $\frac{d}{dx}(4^x \sin x)$ **(3 marks)**

Answer

2 A curve has equation $xy + \left(\frac{1}{2}\right)^y = 2$

Find an equation of the normal to the curve at the point (0, −1). **(6 marks)**

Answer

Use the product rule. Look at page 90 for a reminder.

The gradient of the normal will be $\frac{-1}{\left(\frac{dy}{dx}\right)}$

Small angle approximations

The proof shown above makes use of small angle approximations for sin and cos. When x is small and measured in radians:

$\sin x \approx x \qquad \cos x \approx 1 - \frac{1}{2}x^2 \qquad \tan x \approx x$

You can use these approximations to deduce the limits given in the proof above. When h is small:

$\frac{\sin h}{h} \approx \frac{h}{h} = 1$ and

$\frac{\cos h - 1}{h} \approx \frac{1 - \frac{1}{2}h^2 - 1}{h} = -\frac{1}{2}h$

This tends to 0 as $h \to 0$

Worked example

Show that, for small values of θ measured in radians, $\frac{1 - 4\cos\theta + \sin\theta}{3 + \tan 2\theta} \approx -1$ **(4 marks)**

$$\frac{1 - 4\cos\theta + \sin\theta}{3 + \tan 2\theta} \approx \frac{1 - 4(1 - \frac{1}{2}\theta^2) + \theta}{3 + 2\theta}$$
$$= \frac{2\theta^2 + \theta - 3}{3 + 2\theta}$$
$$= \frac{(2\theta + 3)(\theta - 1)}{2\theta + 3}$$
$$= \theta - 1$$

$\theta - 1 \to -1$ as $\theta \to 0$

If θ is small then you can assume that small multiples of θ such as 2θ and 3θ are also small.

Now try this

1 (a) Given that $f(x) = \sin x$, show that

$$f'(x) = \lim_{h\to 0}\left(\left(\frac{\cos h - 1}{h}\right)\sin x + \left(\frac{\sin h}{h}\right)\cos x\right)$$

Answer **(3 marks)**

(b) Hence prove that $f'(x) = \cos x$ **(2 marks)**

Answer

2 Given that θ is small and measured in radians, show that $\frac{1 - \cos 3\theta}{\tan\theta \sin\theta} \approx \frac{9}{2}$ **(4 marks)**

Answer

3 $f(x) = \cos x$

Prove, from first principles, that $f'\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

(5 marks)

Answer

In question 3, follow the same steps that are shown in the worked example above, but replace x with $\frac{\pi}{3}$.

The product rule

The product rule lets you differentiate two functions **multiplied** together. It's not given in the formulae booklet so you need to **learn** it. Here's an example of a function that can be differentiated using the product rule:

$$y = \underbrace{e^x}_{u}\underbrace{(x^2 - 2)}_{v}$$

Golden rule

If $y = uv$, where u and v are functions of x, then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

In function notation, if $h(x) = f(x)g(x)$, then $h'(x) = f(x)g'(x) + g(x)f'(x)$

Worked example

Differentiate with respect to x

(a) $y = x^4 \ln 3x$ **(3 marks)**

$u = x^4 \qquad v = \ln 3x$

$\frac{du}{dx} = 4x^3 \qquad \frac{dv}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = x^4\left(\frac{1}{x}\right) + (\ln 3x)(4x^3)$

$= x^3(1 + 4\ln 3x)$

(b) $y = e^x(1 + \cos 2x)$ **(3 marks)**

$u = e^x \qquad v = 1 + \cos 2x$

$\frac{du}{dx} = e^x \qquad \frac{dv}{dx} = -2\sin 2x$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = e^x(-2\sin 2x) + (1 + \cos 2x)e^x$

$= e^x(1 - 2\sin 2x + \cos 2x)$

For any constant, k

$\frac{d}{dx}(\ln kx) = \frac{1}{kx} \times k = \frac{1}{x}$

It's easy to get in a mess with the product rule.

Start by writing out u, v, $\frac{du}{dx}$ and $\frac{dv}{dx}$.

Then use brackets when you substitute to make sure you don't make a mistake.

Formulae booklet

Unless you are specifically asked to show them, you can quote these results from the formulae booklet:

$f(x)$	$f'(x)$
$\tan kx$	$k\sec^2 kx$
$\sec kx$	$k\sec kx \tan kx$
$\cot kx$	$-k\,\mathrm{cosec}^2 kx$
$\mathrm{cosec}\, kx$	$-k\,\mathrm{cosec}\, kx \cot kx$

For part (b) you need to look at the factors of $h'(x)$. e^{2x} and $\mathrm{cosec}\, x$ can never equal 0, so the factor $(2 - \cot x)$ must equal 0.

Worked example

(a) Given that $h(x) = e^{2x}\mathrm{cosec}\, x$, find $h'(x)$. **(4 marks)**

$h(x) = f(x)g(x)$

$f(x) = e^{2x} \qquad g(x) = \mathrm{cosec}\, x$

$f'(x) = 2e^{2x} \qquad g'(x) = -\mathrm{cosec}\, x \cot x$

$h'(x) = f(x)g'(x) + g(x)f'(x)$

$= e^{2x}(-\mathrm{cosec}\, x \cot x) + (\mathrm{cosec}\, x)(2e^{2x})$

$= e^{2x}\mathrm{cosec}\, x(2 - \cot x)$

(b) Solve, in the interval $0 \leqslant x < \frac{\pi}{2}$, the equation $h'(x) = 0$. **(4 marks)**

$e^{2x}\mathrm{cosec}\, x(2 - \cot x) = 0$

$e^{2x} \neq 0 \qquad \mathrm{cosec}\, x \neq 0 \qquad 2 - \cot x = 0$

$\tan x = \frac{1}{2}$

$x = 0.464$ (3 s.f.)

Now try this

1 Differentiate with respect to x

(a) $\sqrt{x}\sin 5x$ Answer **(3 marks)**

(b) $\ln x^x$ Answer **(3 marks)**

Write $\ln x^x$ as $x \ln x$

2 A curve has equation $y = e^x(3x^2 - 4x - 1)$

(a) Find $\frac{dy}{dx}$ Answer **(4 marks)**

(b) Find the x-coordinates of the turning points on the curve. Answer **(4 marks)**

The quotient rule

If one function is **divided** by another you can differentiate them using the quotient rule.
One version is given in the formulae booklet:

$$\frac{f(x)}{g(x)} \qquad \frac{f'(x)\,g(x) - f(x)\,g'(x)}{(g(x))^2}$$

function derivative

But the version on the right is often easier to use, so you should **learn** it for your exam.

Golden rule

If $y = \frac{u}{v}$, where u and v are functions of x, then

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

The minus sign on top of the fraction means the order is important.

Worked example

Differentiate $\frac{\sin 5x}{x^2}$ with respect to x. **(3 marks)**

$u = \sin 5x \qquad v = x^2$

$\frac{du}{dx} = 5\cos 5x \qquad \frac{dv}{dx} = 2x$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(x^2)(5\cos 5x) - (\sin 5x)(2x)}{(x^2)^2}$$

$$= \frac{5x\cos 5x - 2\sin 5x}{x^3}$$

When using the quotient rule, start by writing out u, v, $\frac{du}{dx}$ and $\frac{dv}{dx}$. Then use brackets when you substitute to make sure you don't make a mistake.

Make sure you are really confident differentiating expressions like $\sin 5x$ using the chain rule.

This solution shows the quotient rule using function notation. You will have to answer questions about curves or graphs in the form $y = \ldots$ **and** questions about functions in the form $f: x \mapsto \ldots$ so it's a good idea to be comfortable with both versions.

Worked example

The function h is defined by

$$h: x \mapsto \frac{e^x + 2}{e^x - 3}, \quad x \in \mathbb{R}, \quad x \neq \ln 3$$

Show that $h'(x) = \frac{-5e^x}{(e^x - 3)^2}$ **(3 marks)**

$h(x) = \frac{f(x)}{g(x)}$

$f(x) = e^x + 2 \qquad g(x) = e^x - 3$

$f'(x) = e^x \qquad g'(x) = e^x$

$$h'(x) = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2}$$

$$= \frac{e^x(e^x - 3) - (e^x + 2)e^x}{(e^x - 3)^2}$$

$$= \frac{-5e^x}{(e^x - 3)^2}$$

3 (a) $f(x) = \frac{3(x^2+2)}{(x^2+2)(x-1)} - \frac{3(x-1)}{(x^2+2)(x-1)} - \frac{9}{(x^2+2)(x-1)}$

$$= \frac{3(x^2+2) - 3(x-1) - 9}{(x^2+2)(x-1)}$$

$$= \frac{3x\cancel{(x-1)}}{(x^2+2)\cancel{(x-1)}}$$

$$= \frac{3x}{x^2+2}$$

(b) $f'(x) = \frac{(x^2+2)(3) - (3x)(2x)}{(x^2+2)^2}$

$$= \frac{6 - 3x^2}{(x^2+2)^2}$$

(a) $\frac{\cos x}{\sqrt{x}}$ Answer **(3 marks)**

(b) $\frac{2x}{\sqrt{3x+1}}$ Answer **(3 marks)**

(c) $\frac{x^2+1}{\ln(x^2+1)}$ Answer **(4 marks)**

2 Use the quotient rule to show that

$\frac{d}{dx}(\cot x) = -\mathrm{cosec}^2 x$ **(4 marks)** Answer

3 $f(x) = \frac{3}{x-1} - \frac{3}{x^2+2} - \frac{9}{(x^2+2)(x-1)}$

(a) Show that $f(x) = \frac{3x}{x^2+2}$ **(4 marks)** Answer

(b) Hence, or otherwise, find $f'(x)$ in its simplest form. **(3 marks)** Answer

The chain rule

The chain rule lets you differentiate a **function of a function**. The chain rule is not in the formulae booklet, so make sure you know how to use it confidently. Follow this step-by-step method:

1 Substitute the 'inside' function with u.

$y = \sqrt{x^3 + 1}$
$u = x^3 + 1$
$y = \sqrt{u} = u^{\frac{1}{2}}$

2 Treat u as a single variable and differentiate.

$$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2}(x^3+1)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x^3+1}}$$

3 Multiply the result by the derivative of u.

$$\frac{du}{dx} = 3x^2$$

$$\underbrace{\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}}_{\text{This is how the chain rule is sometimes stated.}} = \frac{3x^2}{2\sqrt{x^3+1}}$$

Worked example

$f(x) = \dfrac{2}{3x+1}, \quad x > 0$

Differentiate $f(x)$ and find $f'(1)$. **(3 marks)**

$y = f(x) = 2(3x+1)^{-1}$
$= 2u^{-1}$ with $u = 3x + 1$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -2u^{-2} \times 3 = -6(3x+1)^{-2}$$

$$f'(x) = \frac{-6}{(3x+1)^2} \qquad f'(1) = \frac{-6}{(3 \times 1 + 1)^2} = -\frac{3}{8}$$

You don't have to write down all of this working to apply the chain rule in your exam. If you're confident you can jump straight from the first line of working to:
$f'(x) = -2(3x+1)^{-2} \times 3$

Splodge

You can use the splodge method to apply the chain rule quickly. The 'inside' function is splodge. You work out $\dfrac{dy}{d(\text{splodge})}$, then multiply by the derivative of splodge!

$$y = (5x - 1)^9$$

$$\frac{dy}{dx} = \underbrace{5}_{\text{Derivative of splodge}} \times \underbrace{9(5x-1)^8}_{\frac{dy}{d(\text{splodge})}}$$

Worked example

C is a curve with equation $2y^3 + 10y + 2 = x$

(a) Find $\dfrac{dy}{dx}$ in terms of y. **(3 marks)**

$$\frac{dx}{dy} = 6y^2 + 10 \text{ so } \frac{dy}{dx} = \frac{1}{6y^2 + 10}$$

(b) Find the gradient of the curve at the point $(-8, -2)$. **(2 marks)**

$$y = -2, \text{ so } \frac{dy}{dx} = \frac{1}{6(-2)^2 + 10} = \frac{1}{34}$$

Functions of y

You can use this version of the chain rule to differentiate equations where x is given in terms of y: $\dfrac{dy}{dx} = \dfrac{1}{\left(\dfrac{dx}{dy}\right)}$

$\dfrac{dy}{dx}$ is given in terms of y so you need to substitute the **y-coordinate** to find the gradient.

Now try this

1 Given that $y = \dfrac{1}{\sqrt{x^2 - 3x + 1}}$, find $\dfrac{dy}{dx}$

Answer **(3 marks)**

2 $f(x) = (\sqrt[3]{x} + 6)^6$. Differentiate $f(x)$ and find $f'(8)$, writing your answer as a power of 2.

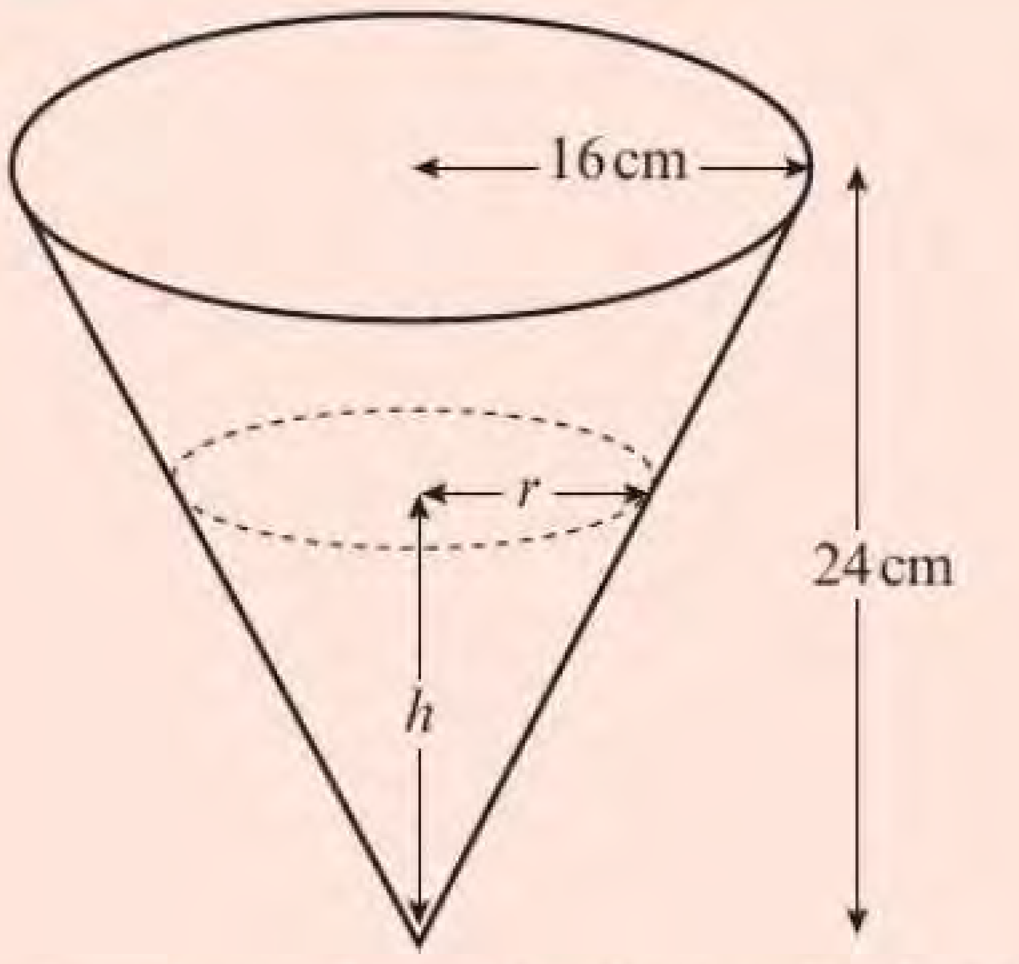
Answer **(4 marks)**

Had a look ☐ Nearly there ☐ Nailed it! ☐

Rates of change

You can model lots of physical or financial situations by describing how a variable changes with time. Equations involving rates of change are called **differential equations**. You can revise how to **form** differential equations on this page, and how to **solve** them on page 109.

Worked example



A container is made in the shape of a hollow inverted right circular cone. The height of the container is 24 cm and the radius is 16 cm, as shown in the diagram. Water is flowing into the container. When the height of water is h cm, the surface of the water has radius r cm and the volume of water is V cm^3.

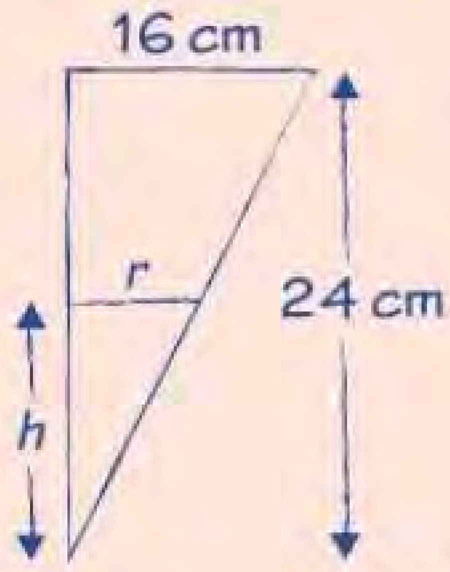
(a) Show that $V = \frac{4\pi h^3}{27}$ **(2 marks)**

[*The volume V of a right circular cone with vertical height h and base radius r is given by the formula* $V = \frac{1}{3}\pi r^2 h$.]

$\frac{r}{h} = \frac{16}{24}$ so $r = \frac{2h}{3}$

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{2h}{3}\right)^2 h$

$= \frac{4\pi h^3}{27}$



Water flows into the container at a rate of 8 cm^3 s^{-1}.

(b) Find, in terms of π, the rate of change of h when $h = 12$. **(5 marks)**

$\frac{dV}{dh} = \frac{12\pi h^2}{27} = \frac{4\pi h^2}{9}$

$\frac{dh}{dt} = \frac{dV}{dt} \div \frac{dV}{dh} = 8 \div \frac{4\pi h^2}{9}$

$= 8 \times \frac{9}{4\pi h^2}$

$= \frac{18}{\pi h^2}$

When $h = 12$:

$\frac{dh}{dt} = \frac{18}{\pi(12)^2} = \frac{1}{8\pi}$

If you need to use the formula for the volume of a cone in your exam it will be given to you with the question. You can use similar triangles to find the relationship between r and h. Draw a sketch to help you.

Chain rule

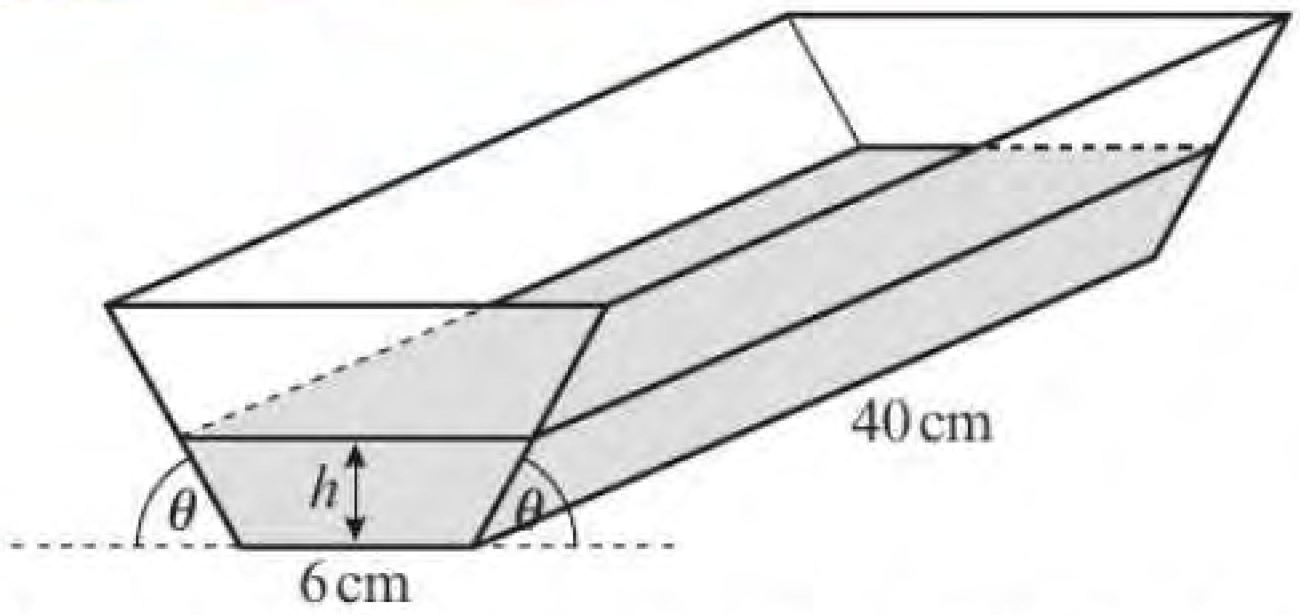
The chain rule allows you to multiply and divide derivatives in the same way as fractions.

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ so $\frac{dh}{dt} = \frac{dV}{dt} \div \frac{dV}{dh}$

Use your answer to part (a) to find $\frac{dV}{dh}$. The rate the water is flowing into the container is $\frac{dV}{dt} = 8$ cm^3 s^{-1}. The rate of change of h is $\frac{dh}{dt}$.

For part (a), remember that θ is a constant.

Now try this



The diagram shows a section of gutter in the shape of a prism. The cross-section of the gutter is a symmetrical trapezium. Water is flowing into the gutter. When the depth of the water is h cm, the volume of the water is V cm^3.

(a) Show that $\frac{dV}{dh} = 240 + 80h\cot\theta$ **(3 marks)**

Answer

Water flows into the gutter at a constant rate of 40 cm^3 s^{-1}.

(b) Given that when $h = 2.5$ the rate of change of h is 0.1 cm s^{-1}, find the value of θ correct to 1 decimal place. **(5 marks)**

Answer