Proof by Contradiction

There's one more type of proof that you need to know for your A Level, and it's a bit of a mind-bender. Very clever though, once you get your head round it...

Proof by Contradiction

To prove a statement by **contradiction**, you say 'Assume the statement is **not true**...', then prove that something **impossible** would have to be true for that to be the case.

Example: Prove the following statement: "If x^2 is even, then x must be even."

We can prove the statement by contradiction.

Assume the statement is **not true**. Then there must be an **odd number** x for which x^2 is **even**.

If x is odd, then you can write x as $2k + 1$, where k is an integer — this is the definition of an odd number.

Now, $x^2 = (2k + 1)^2 = 4k^2 + 4k + 1$

$4k^2 + 4k = 2(2k^2 + 2k)$ is **even** because it is $2 \times$ an integer — this is the definition of an even number.

$\Rightarrow 4k^2 + 4k + 1$ is **odd** — since even + odd = odd.

But this **isn't possible** if the assumption that x^2 is even is true.
We've **contradicted** the assumption that there is an odd number x for which x^2 is even.

So if x^2 is **even**, then x must be **even**, hence the original statement is **true**.

Surds are Irrational

You can use **proof by contradiction** to prove some **really important** facts.
For example, you can prove that the **square root** of **any** non-square number is **irrational**.

Example: Prove that $\sqrt{2}$ is irrational.

We can prove the statement by contradiction.

Start by assuming that the statement is **not true**, and that $\sqrt{2}$ can be written as $\frac{a}{b}$ with a and b both **non-zero integers** — using the definition of a rational number.

You can also assume that a and b do **not** have any **common factors**.

If $\sqrt{2} = \frac{a}{b}$, then $\sqrt{2}b = a$.

Squaring both sides gives you $2b^2 = a^2$ so a^2 is an **even** number — using the definition of an even number.

You saw in the previous example that if a^2 is **even**, then a must be **even** as well.
So replace a with $2k$ for some integer k:

$$2b^2 = (2k)^2 = 4k^2 \Rightarrow b^2 = 2k^2.$$

Like before, this tells you that b must be **even** (since b^2 is even).
However, you assumed at the start that a and b had **no common factors**, so you have **contradicted** your initial assumption.

Therefore $\sqrt{2}$ **cannot** be written as a fraction $\frac{a}{b}$, so it is **irrational**.

You can use the same method to prove the irrationality of **any** surd, although you need to prove the statement "If x^2 is a multiple of a prime number p, then x must also be a multiple of p", which is a bit **trickier** than the proof in the first example above.

Proof by Contradiction

You can prove there are *Infinitely Many* of something

Another use for proof by contradiction is to show that there are **infinitely many** numbers in a **certain set**.

Example: Prove by contradiction that there are infinitely many even numbers.

Assume that there are a **finite** number of even numbers, and that the biggest one can be written N, where $N = 2n$ and n is an integer.

But if you add 2 to this, you get $N + 2 = 2n + 2 = 2(n + 1)$, which is an even number **bigger** than N.

You assumed that N is the biggest even number, so you have **contradicted** your initial assumption.

So there are **infinitely many** even numbers.

You can use the same method to show that there are infinitely many **odd numbers**, or **multiples** of **5**, or **16** or **any number**. To prove that there are infinitely many **prime numbers**, you need a **slightly different** method:

Example: Prove by contradiction that there are infinitely many prime numbers.

Assume that there are a **finite** number of primes (say n), and label them $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, ... , p_{n-1}, p_n, so p_n is the **largest** prime number.

Using subscripts is standard notation for writing lists of any numbers.

Now **multiply** all of these together: $p_1 p_2 p_3 \cdots p_{n-1} p_n$ — call this number P.

Because of how you defined it, P is a **multiple** of **every** prime number.

Now think about $P + 1$ — if you **divide** $P + 1$ by p_1, you get:

$$(P + 1) \div p_1 = (p_1 p_2 p_3 \cdots p_{n-1} p_n + 1) \div p_1$$
$$= p_2 p_3 \cdots p_{n-1} p_n \text{ remainder } 1.$$

In fact, dividing $(P + 1)$ by any prime number gives a **remainder** of **1**.

So $(P + 1)$ **isn't divisible** by **any** of the prime numbers, so either it is **also** a prime number, or it is a **product** of some other prime numbers that aren't in the list.

Either way, there is at least one prime number that is **not** on the list, which **contradicts** the assumption that the list contained all of them, so there must be **infinitely many** prime numbers.

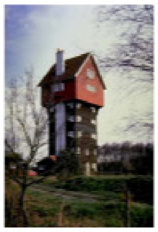
Practice Questions

Q1 Prove by contradiction that if x is irrational, then $-x$ is also irrational.

Q2 Prove by contradiction that there is no largest multiple of 21.

Q3 a) Prove by contradiction that there is no largest integer power of 10.

b) Prove by contradiction that there is no smallest integer power of 10.



Proof by photograph that CGP HQ is irrational.

Exam Questions

Q1 Prove by contradiction that if $p + q$ is irrational, then at least one of p or q is irrational. [3 marks]

Q2 For any two prime numbers p and q, where $p > 2$ and $q > 2$, prove by contradiction that pq is always odd. [3 marks]

Proof that pantomimes are hilarious: Oh no they aren't... Oh. Yes they are...

One crucial point to remember with proofs is that you have to justify every step of your working. Make sure that you've got a mathematical rule or principle to back up each bit of the proof — you can't take anything for granted.