Modulus

The modulus of a number is really useful if you don't care whether something's positive or negative — like if you were more interested in the size than the sign. It pops up in a few different places in A-level maths.

***Modulus** is the **Size** of a number*

1) The **modulus** of a number is its **size** — it doesn't matter if it's **positive** or **negative**. So for a **positive** number, the modulus is just the **same** as the number itself, but for a **negative** number, the modulus is its **positive value**. For example, the modulus of 8 is 8, and the modulus of −8 is also 8.
2) The modulus of a number, x, is written $|x|$. So the example above would be written $|8| = |-8| = 8$.
3) In **general** terms, for $x \geq 0$, $|x| = x$ and for $x < 0$, $|x| = -x$.
4) **Functions** can have a modulus too — the modulus of a function $f(x)$ is its **positive value**. Suppose $f(x) = -6$, then $|f(x)| = 6$. In general terms:

   $|f(x)| = f(x)$ when $f(x) \geq 0$ and
   $|f(x)| = -f(x)$ when $f(x) < 0$.

5) If the modulus is **inside** the brackets in the form $f(|x|)$, then you make the x-value positive **before** applying the function. So $f(|-2|) = f(2)$.

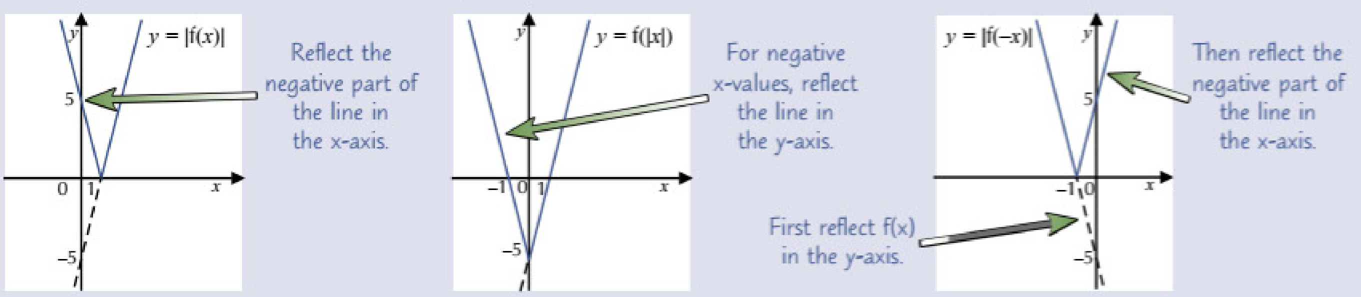
The modulus is sometimes called the absolute value.

Graphs** of **Modulus** functions are **Reflected** in the **Axes

It's likely that you'll have to draw the **graph** of a modulus function — and there are **three different types**.

1) For the graph of $y = |f(x)|$, any **negative** values of $f(x)$ are made **positive** by **reflecting** them in the **x-axis**. This **restricts** the **range** of the modulus function to $|f(x)| \geq 0$ (or some subset **within** $|f(x)| \geq 0$, e.g. $|f(x)| \geq 1$).
2) For the graph of $y = f(|x|)$, the **negative** x-values produce the **same result** as the corresponding **positive** x-values. So the graph of $f(x)$ for $x \geq 0$ is **reflected** in the **y-axis** for the negative x-values.
3) For the graph of $y = |f(-x)|$, the x-values change from **positive to negative** (or **negative to positive**), so the graph is **reflected** in the **y-axis**. Then any **negative** values of $f(x)$ are made **positive** by **reflecting** them in the **x-axis**. As with the graph of $y = |f(x)|$, the **range** is **restricted**.
4) The easiest way to draw these graphs is to draw $f(x)$ (**ignoring** the modulus for now), then **reflect** it in the **appropriate axis** (or **axes**). This will probably make more sense when you've had a look at an **example**:

Example: Draw the graphs of $y = |f(x)|$, $y = f(|x|)$ and $|f(-x)|$ for the function $f(x) = 5x - 5$.



***Solving** modulus functions usually produces **More Than One** solution*

Here comes the method for solving '$|f(x)| = n$'. Solving '$|f(x)| = g(x)$' is **exactly the same** — just replace n with $g(x)$.

Solving Modulus Equations of the form $|f(x)| = n$

1) First, **sketch** the functions $y = |f(x)|$ and $y = n$, on the **same** axes.
2) From the graph, work out the **ranges of x** for which $f(x) \geq 0$ and $f(x) < 0$:
   E.g. $f(x) \geq 0$ for $x \leq a$ or $x \geq b$ and $f(x) < 0$ for $a < x < b$.
3) Use this to write two **new equations**, one true for each range of x:
   (1) $f(x) = n$ for $x \leq a$ or $x \geq b$ (2) $-f(x) = n$ for $a < x < b$
4) Finally, just **solve** each equation and check that any solutions are valid — get rid of any solutions outside the range of x you've got for that equation. You can also look back at the graph to **check** that your solutions look right.

The solutions you're trying to find are where the functions intersect.

These ranges should 'fit together' to cover all possible values of x.

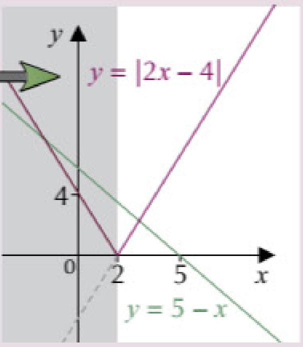
Modulus

Sketch the Graph to see *How Many Solutions* there are

Okay, so that method probably sounds complicated, but it really makes a lot more sense when you see it in action.

Example: Solve $|2x-4| = 5-x$. This is an example of $|f(x)| = g(x)$, where $f(x) = 2x-4$ and $g(x) = 5-x$.

First off, sketch the graphs of $y = |2x-4|$ and $y = 5-x$.
They cross at 2 different points, so there should be 2 solutions.



Now find out where $f(x) \geq 0$ and $f(x) < 0$:
$2x-4 \geq 0$ for $x \geq 2$, and $2x-4 < 0$ for $x < 2$ (shaded).

Form two equations for the different ranges of x:
(1) $2x-4 = 5-x$ for $x \geq 2$ (2) $-(2x-4) = 5-x$ for $x < 2$

Solving (1) gives: $3x = 9 \Rightarrow$ $\boldsymbol{x = 3}$ ← valid because $3 \geq 2$

Solving (2) gives: $-x = 1 \Rightarrow$ $\boldsymbol{x = -1}$ ← valid because $-1 < 2$

Check back against the graphs — we've found two solutions and they're in the right places. Nice.

You can also *Solve* modulus equations *Algebraically*

Using the graphical method isn't too bad for equations of the form $|f(x)| = g(x)$, but when you're faced with something like $|f(x)| = |g(x)|$, it can get pretty complicated (you've got to think about where $f(x)$ is positive and negative, **and** where $g(x)$ is positive and negative — total hassle).
Fortunately there's also an **algebraic** method you can use:

If $|a| = |b|$ then $a^2 = b^2$.
So if $|f(x)| = |g(x)|$ then $[f(x)]^2 = [g(x)]^2$.

Example: Solve $|x-2| = |3x+4|$.

Start by squaring both sides:
$$|x-2| = |3x+4|$$
$$(x-2)^2 = (3x+4)^2$$
Now expand and simplify:
$$x^2 - 4x + 4 = 9x^2 + 24x + 16$$
$$8x^2 + 28x + 12 = 0$$
$$2x^2 + 7x + 3 = 0$$
$$(2x+1)(x+3) = 0$$
So the solutions are: $x = -\frac{1}{2}$ and $x = -3$

You can also use these methods to solve **modulus inequalities**. Since you often end up **solving a quadratic** in this kind of question, the **graphical method** from p.24 is pretty darn useful. Another useful rule is that:
$|x-a| < b \Leftrightarrow a-b < x < a+b$

Practice Questions

Q1 a) For the function $f(x) = 2x-1$, $x \in \mathbb{R}$, sketch the graphs of:
(i) $y = |f(x)|$ (ii) $y = f(|x|)$

b) Hence, or otherwise, solve the equation $|2x-1| = 5$.

Q2 Find the range of values of x that satisfy:
a) $|x| < 4$ b) $|2x| > 12$ c) $|x+3| \leq 3$

Exam Questions

Q1 Solve the equation $3|-x-6| = x+12$. [3 marks]

Q2 a) Show that the equation $|2x+1| = |x-k|$ can be transformed into the quadratic equation $3x^2 + (4+2k)x + (1-k^2) = 0$. [3 marks]

b) Hence find the value(s) of k for which $|2x+1| = |x-k|$ has exactly one solution. [4 marks]

My name is Modulus Functionas Meridius...

So if the effect of the modulus is to make a negative positive, I guess that means that |exam followed by detention followed by getting splashed by a car| = sleep-in followed by picnic followed by date with Hugh Jackman. I wish.