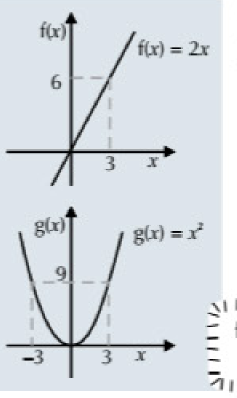
Composite and Inverse Functions

A mapping takes one number and transforms it into another — e.g. 'multiply by 5', 'square root' and 'divide by 7' are all mappings. Sadly they have more to do with functions than drawing a map with rivers, mountains and secret tunnels.

*A **Function** is a type of **Mapping***

1) A **function** is an operation that takes numbers and **maps** each one to only one number — e.g. x^2 is written $f(x) = x^2$ or $f : x \to x^2$.
2) The set of starting numbers is the **domain** and the numbers they become is the **range** (or **image**).
3) The domain and/or range is often the set of **real numbers**, $\mathbb{R}$ (any number, fraction, surd etc.). If x can take **any real value** it's usually written $x \in \mathbb{R}$.
4) A **one-to-one** function maps **one** element in the **domain** to **one** element in the **range**.
5) A **many-to-one** function maps **more than one** element in the **domain** to **one** element in the **range**.
6) **One-to-many** mappings can take an element in the domain to **more than one** element in the range (e.g. 'take the square root' could map 1 to +1 or –1) — so by definition, they're **not functions**.



$f(x) = 2x$ and $g(x) = x^2$ are functions because each value of x can only be one value of f(x) or g(x).

Both functions have domain $x \in \mathbb{R}$. But f(x) has range $f(x) \in \mathbb{R}$ and g(x) has range $g(x) \geq 0$.

f(x) is a one-to-one function and g(x) is a many-to-one function.

*Functions can be **Combined** to make a **Composite Function***

1) If you have two functions f and g, you can **combine** them (do one followed by the other) to make a new function. This is called a **composite function**.
2) Composite functions are written $fg(x)$ — this means do g first, then f. **Brackets** can be handy here, so $fg(x) = f(g(x))$. The **order** is really important — usually $fg(x) \neq gf(x)$.
3) If you get a composite function that's written $f^2(x)$, it means $ff(x)$ — you do f **twice**.

Composite functions made up of more than two functions work in the same way.

Example: For the functions $f : x \to 2x^3$ $\{x \in \mathbb{R}\}$ and $g : x \to x - 3$ $\{x \in \mathbb{R}\}$, find:
a) $fg(4)$ b) $gf(4)$ c) $fg(x)$ d) $f^2(x)$.

From parts a) and b) you can see that $fg(4) \neq gf(4)$.

a) $fg(4) = f(g(4)) = f(4 - 3) = f(1) = 2 \times 1^3 = 2$

b) $gf(4) = g(f(4)) = g(2 \times 4^3) = g(128) = 128 - 3 = 125$

c) $fg(x) = f(g(x)) = f(x - 3) = 2(x - 3)^3$

d) $f^2(x) = f(f(x)) = f(2x^3) = 2(2x^3)^3 = 16x^9$

*You could be asked to **Solve** a **Composite Function Equation***

If you're asked to **solve** an equation involving a composite function, such as $fg(x) = 8$, start by finding $fg(x)$.

Example: For the functions $f : x \to \sqrt{x}$, domain $\{x \geq 0\}$ and $g : x \to \frac{1}{x-1}$, domain $\{x > 1\}$, solve $fg(x) = \frac{1}{2}$ and state the range of $fg(x)$, giving your answer in set notation.

$\sqrt{x}$ means the positive root, so f is a function.

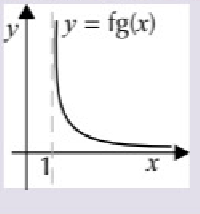
First, find fg(x): $fg(x) = f\left(\frac{1}{x-1}\right) = \sqrt{\frac{1}{x-1}} = \frac{1}{\sqrt{x-1}}$

So $\frac{1}{\sqrt{x-1}} = \frac{1}{2}$

Rearrange this equation to find x:

$\frac{1}{\sqrt{x-1}} = \frac{1}{2} \Rightarrow \sqrt{x-1} = 2 \Rightarrow x - 1 = 4 \Rightarrow x = 5$

You can see the range of fg(x) from the graph.



The range is $fg(x) > 0$. In set notation, that's $\{fg(x) : fg(x) > 0\}$

The domain of fg(x) is $x > 1$ or $\{x : x > 1\}$.

*Only **One-to-One Functions** have **Inverses***

1) An **inverse function** does the **opposite** to the function. For a function f(x), the inverse is written $\mathbf{f^{-1}(x)}$.
2) An inverse function **maps** an element in the **range** to an element in the **domain** — the opposite of a function. This means that only **one-to-one functions** have inverses, otherwise it wouldn't be a function by definition.
3) For **any** inverse $f^{-1}(x)$, doing the function and then the inverse is the same as doing the inverse then doing the function — both just give you x.

$f^{-1}f(x) = x = ff^{-1}(x)$

4) The **domain** of the **inverse** is the **range** of the **function**, and the **range** of the **inverse** is the **domain** of the **function**.

Composite and Inverse Functions

Work out the *Inverse Function* using *Algebra*

For **simple** functions it's easy to work out what the inverse is just by **looking** at it — e.g. $f(x) = x + 1$ has the inverse $f^{-1}(x) = x - 1$. But for more **complex** functions, you need to **rearrange** the original function to **change** the **subject**.

Example: Find the inverse of the function $f(x) = 3x^2 + 2$ with domain $x \geq 0$, and state its domain and range.

1) First, replace f(x) with y — its easier to work with than f(x):

$y = 3x^2 + 2$

2) Rearrange the equation to make x the subject:

$y - 2 = 3x^2 \Rightarrow \frac{y-2}{3} = x^2 \Rightarrow \sqrt{\frac{y-2}{3}} = x$

$x \geq 0$ so you don't need the negative square root.

3) Replace x with $f^{-1}(x)$ and y with x:

$\mathbf{f^{-1}(x) = \sqrt{\frac{x-2}{3}}}$

4) Swap the domain and range:

The range of $f(x)$ is $f(x) \geq 2$,
so $f^{-1}(x)$ has domain $x \geq 2$ and range $f^{-1}(x) \geq 0$.

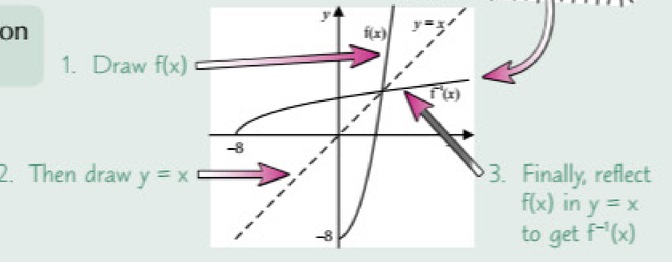
You might have to *Draw the Graph* of the *Inverse*

The inverse of a function is its **reflection** in the line $y = x$.

The inverse function is $f^{-1}(x) = \sqrt{x+8}$.

Example: Sketch the graph of the inverse of the function $f(x) = x^2 - 8$ with domain $x \geq 0$.

It's easy to see what the domains and ranges are from the graph — $f(x)$ has domain $x \geq 0$ and range $f(x) \geq -8$, and $f^{-1}(x)$ has domain $x \geq -8$ and range $f^{-1}(x) \geq 0$.



Practice Questions

Q1 For each pair of functions f and g, find fg(2), gf(1) and fg(x).

a) $f(x) = \frac{3}{x}$, $x > 0$ and $g(x) = 2x + 3$, $x \in \mathbb{R}$

b) $f(x) = 3x^2$, $x \geq 0$ and $g(x) = x + 4$, $x \in \mathbb{R}$

Q2 A one-to-one function f has domain $x \in \mathbb{R}$ and range $f(x) \geq 3$.
Does this function have an inverse? If so, state its domain and range.

Q3 Using algebra, find the inverse of the function $f(x) = \sqrt{2x-4}$, $x \geq 2$.
State the domain and range of the inverse in set notation.

Instead of $\mathbb{R}$ you might see other sets of numbers such as $\mathbb{Z}$, the set of integers or $\mathbb{N}$, the set of natural numbers (positive integers, not including 0).

Exam Questions

Q1 The functions f and g are given by: $f(x) = x^2 - 3$, $x \in \mathbb{R}$ and $g(x) = \frac{1}{x}$, $x \in \mathbb{R}, x \neq 0$.

a) Find an expression for gf(x). [2 marks]

b) Solve $gf(x) = \frac{1}{6}$ [1 mark]

Q2 The function f(x) is defined as follows: $f : x \rightarrow \frac{1}{x+5}$, domain $x > -5$.

a) State the range of f(x). [1 mark]

b) (i) Find the inverse function, $f^{-1}(x)$. [3 marks]

(ii) State the domain and range of $f^{-1}(x)$. [2 marks]

c) On the same axes, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$. [2 marks]

Inverses — putting the 'nuf' in functions since 1877...

Sorry for making functions more confusing than what you're used to — it's the exam board's fault, honestly. Just make sure you know how to find the domain and range and understand what the notation means. The rest of it isn't so bad, but have a good practice anyway — functions definitely fall into the 'completely forgot what that means' category.

Modelling and Problem Solving

Modelling and problem solving are two of the three overarching themes of the A-level Maths course (the third being proof, which is covered in Section 1). This means that they could come up in exam questions on any topic.

A **Mathematical Model** simplifies a **Real-life Situation**

A **mathematical model** is a mathematical description of a real-life situation. Modelling involves **simplifying** the situation so that you can understand its behaviour and predict what is going to happen.

Modelling in maths generally boils down to using an **equation** or a **set of equations** to predict what will happen in real life. You'll meet it in **all** areas of this course including population growth in **algebra** (see pages 82-83), moving objects in **mechanics** (see Section 16) and probability distributions in **statistics** (see Section 13).

Models use **Assumptions**

Models are always **simplifications** of the real-life situation. When you construct a model, you have to make **assumptions**. In other words, you **ignore** or **simplify** some factors that might affect the real-life outcome, in order to keep the maths simpler. For example:

- A population growth model might ignore the fact that the population will eventually run out of **food**, because that won't happen in the **time period** you're modelling.
- A model for the speed of a moving object might ignore **air resistance**, because that would make the maths much **more complicated**, or because you might only want a **general result** for objects of all shapes and sizes.
- Probability distributions based on past data often assume the **conditions** in future trials will be the **same** as when the past data was recorded.

There are lots of special terms to describe the assumptions you might make in mechanics — there's a list of them on page 198.

Example: Leon owns a gooseberry farm. This week, he had 5 workers picking fruit, and they picked a total of 1000 punnets of gooseberries. Leon wants to hire more workers for next week. He predicts that next week, if the number of workers on his farm is w, the farm will produce p punnets of gooseberries, where $p = 200w$.
Suggest three assumptions Leon has made in his model.

This is a model because it is a prediction of how many punnets will be produced — the actual number could be higher or lower.
The model predicts that the average number of punnets produced per worker each week will be the same. For example:

- There will be enough gooseberries to fill 200 punnets per worker, however many workers he employs.
- The weather is good enough to allow each worker to work the same number of hours each week.
- Any new workers he employs will work at the same speed, on average.

There are lots more possible answers here.

You might have to **Criticise** or **Refine** a model

An important part of the modelling process is **refining** a model. This usually happens after the model has been **tested** by comparing it with real-world outcomes, or if you find out some **extra information** that affects the model. Refining a model usually means changing some of the **assumptions**. For example:

- You might adjust a population growth model if you found that **larger populations** were more susceptible to **disease**, so grew more slowly.
- You might decide to refine a model for the speed of an object to take into account the **friction** from the surface the object is travelling over.
- You might adjust a probability distribution if you collect **more data** which changes the **relative frequency** of the outcomes.

You could be asked to criticise or evaluate a model — e.g. you might need to assess if any assumptions are unrealistic.

Example: (cont.) Leon discovers that the weather forecast for next week is bad, and his workers are only likely to be able to pick gooseberries for half the number of hours they did this week. How should the model be refined?

If the workers can only pick for half the time, they'll probably pick half as many gooseberries.

The refined model would be $p = 200w \div 2 \Rightarrow \boldsymbol{p = 100w}$.

Modelling and Problem Solving

Problem Solving questions are more Challenging

Some maths questions can be straightforward to answer — you're told what maths you need to use, then you use it to get a solution. '**Problem solving**' questions are those tricky ones where you have to work out for yourself exactly what maths you need to do.

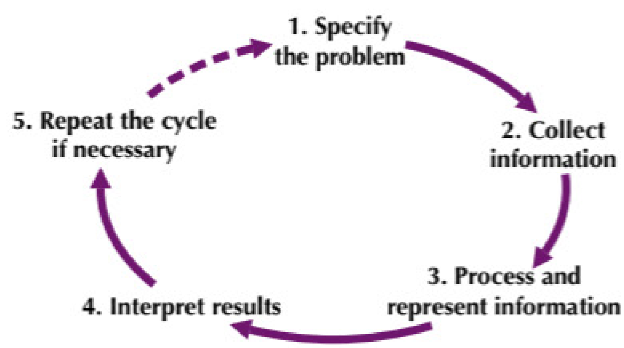
Problem solving questions include:

- questions that don't have '**scaffolding**' (i.e. they're not broken down into parts a), b), c) etc.),
- questions where the information is **disguised** (e.g. a '**wordy**' context, or a **diagram**),
- questions that need more than one **area of maths**,
- questions that test if you actually **understand** the maths as well as being able to use it.

The Problem Solving Cycle can be Useful for maths questions

When it's not obvious what you're supposed to do with a question, you can use the **problem solving cycle**. This breaks the problem up into the following steps:

1. **Specify the problem**
   The first thing to do is work out what the question is actually asking. The question might be phrased in an unusual way or it might be written in a 'wordy' context, where you need to turn the words into maths.
2. **Collect information**
   Write down what you know. All the information you need to answer the question will either be given in the question somewhere (possibly on a diagram), or it'll require facts that you should already know.
3. **Process and represent information**
   When you know what you're trying to find out, and what you already know, you can do the calculation to answer the question.
4. **Interpret results**
   Don't forget to give your answer in terms of the original context. The result of your calculation won't necessarily be the final answer.
5. **Repeat (if necessary)**



When you're doing an exam question, it's unlikely that you'll need to repeat the problem solving cycle once you've calculated the answer — just be aware that it's part of the general problem solving process.

You could also be asked to evaluate the accuracy or limitations of your solutions.

Example: Armand cuts out a semicircle from a rectangular sheet of cardboard measuring 20 cm by 40 cm and throws the rest away. The cardboard he throws away has an area of 398.08 cm². How long is the straight side of the semicircle?

1) What are you trying to find? The length of the straight side of a semicircle is the diameter of the circle, which is twice the radius.

2) What do you know? The total area of the sheet of cardboard is 20 cm × 40 cm. 398.08 cm² was thrown away, so the rest is the area of the semicircle. The area of a semicircle = $\frac{1}{2}$ × area of a circle = $\frac{1}{2}\pi r^2$.

3) Do the maths.

$$\text{Area of semicircle} = (20 \times 40) - 398.08 = 800 - 398.08 = 401.92 \text{ cm}^2$$

$$\text{So: } 401.92 = \frac{1}{2}\pi r^2$$

$$\Rightarrow r^2 = 401.92 \times 2 \div \pi = 803.84 \div \pi \Rightarrow r = \sqrt{803.84 \div \pi} \text{ cm}$$

$$\Rightarrow d = 2r = 2 \times \sqrt{803.84 \div \pi} = 31.99... = 32.0 \text{ cm (3 s.f.)}$$

4) Give the answer in the context of the question. The length of the straight side of the semicircle is **32.0 cm** (3 s.f.).

99% of modelling jobs require A-level maths...

You can apply the problem solving cycle to all sorts: 1. Need to pass A-level Maths exams. 2. Buy CGP revision guide. 3. Knuckle down and get revising. 4. Do some questions and check your answers. 5. Get the kettle on and repeat.*