Partial Fractions

Partial fractions aren't mega useful, but they can be helpful with binomial expansions and integration — welcome back if you're reading Section 4 or 8 and have completely forgotten about them. That's what revision is all about.

'*Expressing in Partial Fractions*' is *Splitting* the fraction up

You can **split** a fraction with **more than one linear factor** in the denominator into **partial fractions**.

$\frac{7x-7}{(2x+1)(x-3)}$ can be written as partial fractions of the form $\frac{A}{(2x+1)}+\frac{B}{(x-3)}$.

$\frac{4}{(x+2)(2x-1)(x-3)}$ can be written as partial fractions of the form $\frac{A}{(x+2)}+\frac{B}{(2x-1)}+\frac{C}{(x-3)}$.

$\frac{3x-8}{(x+2)^2(3x-1)}$ can be written as partial fractions of the form $\frac{A}{(x+2)^2}+\frac{B}{(x+2)}+\frac{C}{(3x-1)}$

Watch out here — this one doesn't quite follow the pattern.

For A-level maths the **numerator** of the fraction will always be **linear** or a **constant**.
You can find A, B and C by using the **substitution method** or the **equating coefficients method**:

Example: Express $\frac{7x-1}{(x-3)(x-1)(x+2)}$ as partial fractions.

Write it out as partial fractions with unknown constants and put it over a common denominator:

$$\frac{7x-1}{(x-3)(x-1)(x+2)} \equiv \frac{A}{(x-3)}+\frac{B}{(x-1)}+\frac{C}{(x+2)}$$
$$\equiv \frac{A(x-1)(x+2)+B(x-3)(x+2)+C(x-3)(x-1)}{(x-3)(x-1)(x+2)}$$

Cancel the denominators from both sides:

$$7x-1 \equiv A(x-1)(x+2)+B(x-3)(x+2)+C(x-3)(x-1)$$

Substitution Method

Substitute values of x which make one of the expressions in brackets equal zero to get rid of all but one of A, B and C:

Substituting $x = 3$ gets rid of B and C
$21 - 1 = A(3-1)(3+2) + 0 + 0$
$20 = 10A \Rightarrow$ $\mathbf{A = 2}$

Substituting $x = 1$ gets rid of A and C:
$7 - 1 = 0 + B(1-3)(1+2) + 0$
$6 = -6B \Rightarrow$ $\mathbf{B = -1}$

Substituting $x = -2$ gets rid of A and B:
$-14 - 1 = 0 + 0 + C(-2-3)(-2-1)$
$-15 = 15C \Rightarrow$ $\mathbf{C = -1}$

OR...

Equating Coefficients Method

Equating coefficients in the numerator will give equations, which you can solve simultaneously to find A, B and C:

Compare coefficients in the numerators:
$7x - 1 \equiv A(x-1)(x+2) + B(x-3)(x+2) + C(x-3)(x-1)$
$\equiv A(x^2+x-2) + B(x^2-x-6) + C(x^2-4x+3)$
$\equiv (A+B+C)x^2 + (A-B-4C)x + (-2A-6B+3C)$

Equating x^2 coefficients: $0 = A + B + C$

Equating x coefficients: $7 = A - B - 4C$

Equating constant terms: $-1 = -2A - 6B + 3C$

Solving these equations simultaneously gives:
$\mathbf{A = 2}$, $\mathbf{B = -1}$ and $\mathbf{C = -1}$.

Finally, replace A, B and C in the original identity:

$$\frac{7x-1}{(x-3)(x-1)(x+2)} \equiv \frac{2}{(x-3)}-\frac{1}{(x-1)}-\frac{1}{(x+2)}$$

Watch out for *Difference of Two Squares* Denominators

Just for added meanness, they might give you an expression like $\frac{4}{x^2-1}$ and tell you to express it as partial fractions.
You have to recognise that the denominator is a **difference of two squares** and write it as **two linear factors**.

Example: Express $\frac{12x+6}{4x^2-9}$ as partial fractions.

The denominator can be factorised: $4x^2 - 9 = (2x+3)(2x-3)$, so write it out as partial fractions and cancel the denominators from both sides:

$$\frac{12x+6}{(2x+3)(2x-3)} \equiv \frac{A}{(2x+3)}+\frac{B}{(2x-3)}$$
$$12x + 6 \equiv A(2x-3) + B(2x+3)$$

Substitute values of x to find A and B:

$x = \frac{3}{2} \Rightarrow 24 = 6B \Rightarrow$ $\mathbf{B = 4}$

$x = -\frac{3}{2} \Rightarrow -12 = -6A \Rightarrow$ $\mathbf{A = 2}$

You could compare coefficients here, but you would need to solve some simultaneous equations.

Replace A and B in the original identity:

$$\frac{12x+6}{(2x+3)(2x-3)} \equiv \frac{2}{(2x+3)}+\frac{4}{(2x-3)}$$

Partial Fractions

*Sometimes it's best to use **Substitution** AND **Equate Coefficients***

Now things are hotting up in the partial fractions department — here's an example involving a **repeated factor**.

Example: Express $\frac{5x+12}{x^2(x-3)}$ in partial fractions.

x is a repeated factor so the identity will be:
$$\frac{5x+12}{x^2(x-3)} \equiv \frac{A}{x^2} + \frac{B}{x} + \frac{C}{(x-3)}$$
$$\equiv \frac{A(x-3) + Bx(x-3) + Cx^2}{x^2(x-3)}$$

Cancel the denominators from both sides:
$$5x + 12 \equiv A(x-3) + Bx(x-3) + Cx^2$$

Substitute values of x to find A and C:
$$x = 0 \Rightarrow 12 = -3A \Rightarrow \mathbf{A = -4}$$
$$x = 3 \Rightarrow 27 = 9C \Rightarrow \mathbf{C = 3}$$

There's no value of x you can substitute to get rid of A and C to leave just B, so equate coefficients of x^2:

Equating x^2 coefficients: $0 = B + C$
You know $C = 3$, so $0 = B + 3 \Rightarrow \mathbf{B = -3}$

Replace A, B and C in the original identity:
$$\frac{5x+12}{x^2(x-3)} \equiv -\frac{4}{x^2} - \frac{3}{x} + \frac{3}{(x-3)}$$

Practice Questions

Q1 Find the values of the constants A and B in the identity $\frac{2x-1}{x^2-x-12} \equiv \frac{A}{x-4} + \frac{B}{x+3}$.

Q2 Express the following as partial fractions:

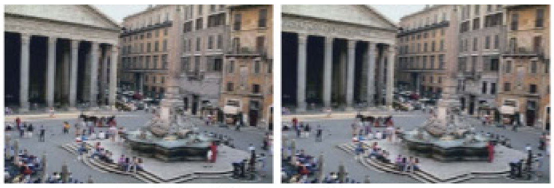
a) $\frac{4x+5}{(x+4)(2x-3)}$ b) $\frac{-7x-7}{(3x+1)(x-2)}$

c) $\frac{x-18}{(x+4)(3x-4)}$ d) $\frac{5x}{x^2+x-6}$

Q3 Express the following as partial fractions:

a) $\frac{2x+2}{(x+3)^2}$ b) $\frac{-18x+14}{(2x-1)^2(x+2)}$

c) $\frac{3x}{(x-5)^2}$ d) $\frac{2x-1}{(x+2)^2(x-3)}$



Spot the difference of two squares (as in, there are 2^2 to find).

Q4 Write $\frac{3x-4}{x^3-16x}$ in the form $\frac{A}{x} + \frac{B}{x+4} + \frac{C}{x-4}$.

Exam Questions

Q1 Given that, for $x \neq -\frac{1}{3}$, $\frac{5+9x}{(1+3x)^2} \equiv \frac{A}{(1+3x)^2} + \frac{B}{(1+3x)}$, where A and B are integers, find the values of A and B. [2 marks]

Q2 Express $\frac{x+4}{(x-2)(x^2-1)}$ as partial fractions. [3 marks]

Q3 Express $\frac{2x-9}{x(x-6)^2}$ as partial fractions. [3 marks]

Q4 a) Use the Factor Theorem to fully factorise $f(x) = x^3 + 5x^2 - x - 5$. [3 marks]

b) Hence write $\frac{3x+1}{x^3+5x^2-x-5}$ as partial fractions. [3 marks]

The algebra was a little dry but I was quite partial to the fractions...

It's worth learning both methods for finding the values of A, B, C etc. Sometimes one's easier to use than the other, and sometimes you might want to mix and match. Just remember the number of factors on the bottom tells you how many fractions you need and a squared term appears in two partial fractions — once squared and once just as it is.