Parametric Equations

Parametric equations seem a bit weird to start with, but they're actually pretty clever. You can use them to replace one horrifically complicated equation with two fairly normal-looking ones. I bet that's just what you always wanted...

Parametric Equations split up x and y into Separate Equations

Normally, graphs in the (x, y) plane are described using a **Cartesian equation** — a single equation linking x and y. Sometimes, particularly for more **complicated** graphs, it's easier to have two linked equations, called **parametric equations**.

In parametric equations, x and y are each **defined separately** in terms of a **third variable**, called a **parameter**. The parameter is usually either t or θ.

Example: Sketch the graph given by the parametric equations $x = t + 1$ and $y = t^2 - 1$.

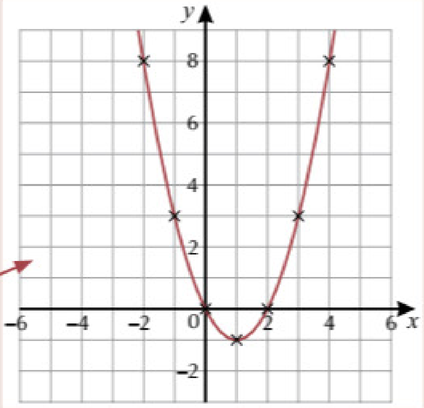
Start by making a **table of coordinates**.
Choose some values for t and calculate x and y at these values.

t	–3	–2	–1	0	1	2	3
x	–2	–1	0	1	2	3	4
y	8	3	0	–1	0	3	8

$x = -2 + 1 = -1$, and $y = (-2)^2 - 1 = 3$

$x = 1 + 1 = 2$, and $y = 1^2 - 1 = 0$

Now plot the **Cartesian (x, y) coordinates** on a set of axes as usual.



You won't necessarily be expected to sketch a curve from its parametric equations in the exam — but finding the Cartesian coordinates like this can make it easier to picture the graph.

You can use the parametric equations of a graph to find **coordinates** of points on the graph, and to find the value of the **parameter** for given ***x*- or *y*-coordinates**.

Example: A curve is defined by the parametric equations $x = 2t - 3$ and $y = \frac{1}{3t}$, $t \neq 0$.

a) Find the x- and y- values of the point the curve passes through when $t = 4$.

b) What value of t corresponds to the point where $y = 9$?

c) What is the value of y when $x = -15$?

Nothing to this question — just sub the right values into the right equations and you're away:

a) When $t = 4$, $x = 8 - 3 = 5$, and $y = \frac{1}{12}$

b) $9 = \frac{1}{3t} \Rightarrow t = \frac{1}{27}$

c) $-15 = 2t - 3 \Rightarrow t = -6 \Rightarrow y = -\frac{1}{18}$

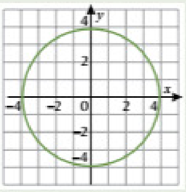
Use the equation for x to find t first, then use that value of t in the other equation to find y.

Circles can be given by Parametric Equations too

You saw the **Cartesian equations** of **circles** on page 38, but now you get to use their **parametric equations**.
A circle with **centre (0, 0)** and **radius r** is defined by the parametric equations $\boldsymbol{x = r\cos\theta}$ and $\boldsymbol{y = r\sin\theta}$ and a circle with **centre (a, b)** and **radius r** is defined by the parametric equations $\boldsymbol{x = r\cos\theta + a}$ and $\boldsymbol{y = r\sin\theta + b}$.

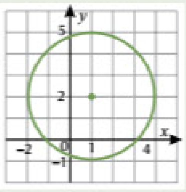
Examples: Sketch the graph given by the equations $x = 4\cos\theta$ and $y = 4\sin\theta$.

This is a circle with radius 4 and centre (0, 0).



Sketch the graph given by the equations $x = 3\cos\theta + 1$ and $y = 3\sin\theta + 2$.

This is a circle with radius 3 and centre (1, 2).



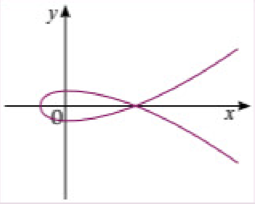
If the values of r are **different** in each equation, you'll get an **ellipse** instead of a circle.
For example, the ellipse given by $x = 2\cos\theta$ and $y = 3\sin\theta$ will be 4 units wide and 6 units tall.

Parametric Equations

*Use **Parametric Equations** to find where graphs **Intersect***

You might have to find points where a parametric equation **intersects** another line.

Example: The curve shown in the sketch on the right has the parametric equations $x = 4t^2 - 1$ and $y = t^3 - t$.
Find the coordinates of the points where the graph crosses:
a) the x-axis,   b) the line $8y = 3x + 3$.



Part a) is pretty straightforward. You've got the y-coordinates already:

a) On the x-axis, $y = 0$.

Use the parametric equation for y to find the values of t where the graph crosses the x-axis:

So: $0 = t^3 - t \Rightarrow t(t^2 - 1) = 0$
$\Rightarrow t(t + 1)(t - 1) = 0$
$\Rightarrow t = 0, t = -1, t = 1$

Now use those values to find the x-coordinates:

$t = 0 \Rightarrow x = 4(0)^2 - 1 = -1$
$t = -1 \Rightarrow x = 4(-1)^2 - 1 = 3$
$t = 1 \Rightarrow x = 4(1)^2 - 1 = 3$

t = –1 and t = 1 give the same coordinates — that's where the curve crosses over itself.

The sketch shows there are two points where the graph crosses each axis.

So the graph crosses the x-axis at the points **(–1, 0)** and **(3, 0)**.

Part b) is just a little trickier. First, sub the parametric equations into 8y = 3x + 3:

b) $8y = 3x + 3 \Rightarrow 8(t^3 - t) = 3(4t^2 - 1) + 3$

Rearrange and factorise to find the values of t you need:

$\Rightarrow 8t^3 - 8t = 12t^2 \Rightarrow 8t^3 - 12t^2 - 8t = 0$
$\Rightarrow 4t(2t + 1)(t - 2) = 0 \Rightarrow t = 0, t = -\frac{1}{2}, t = 2$

Go back to the parametric equations to find the x- and y-coordinates:

$t = 0 \Rightarrow x = -1, y = 0$
$t = -\frac{1}{2} \Rightarrow x = 4\left(-\frac{1}{2}\right)^2 - 1 = 0, y = \left(-\frac{1}{2}\right)^3 + \frac{1}{2} = \frac{3}{8}$
$t = 2 \Rightarrow x = 4(2)^2 - 1 = 15, y = 2^3 - 2 = 6$

So the graph crosses the line $8y = 3x + 3$ at the points **(–1, 0)**, $\left(\mathbf{0}, \frac{3}{8}\right)$, **(15, 6)**.

You can check the answers by sticking these values back into 8y = 3x + 3.

To find the points where the graph crosses the y-axis, you'd find the values of t where x = 0, then use them find the corresponding y-values.

Practice Questions

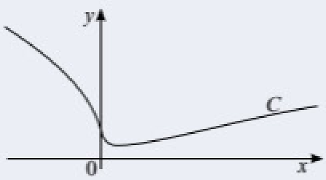
Q1 A curve is defined by the parametric equations $x = \frac{6 - t}{2}$ and $y = 2t^2 + t + 4$.
 a) Find the values of x and y when $t = 0$, 1, 2 and 3.
 b) What are the values of t when: (i) $x = -7$, (ii) $y = 19$?

Q2 For the following circles, write down the radius and the coordinates of the centre:
 a) The circle given by the parametric equations $x = 7 \cos \theta$ and $y = 7 \sin \theta$.
 b) The circle given by the parametric equations $x = 5 \cos \theta + 2$ and $y = 5 \sin \theta - 1$.

Q3 A curve has parametric equations $x = t^2 - 1$ and $y = 4 + \frac{3}{t}$, $t \neq 0$.
What are the coordinates of the points where this curve crosses: a) the y-axis, b) the line $x + 2y = 14$?

Exam Question

Q1 Curve C has parametric equations $x = t^3 + t$, $y = t^2 - 2t + 2$.
 a) K is a point on C, and has the coordinates $(a, 1)$. Find the value of a. [2 marks]
 b) The line $8y = x + 6$ passes through C at points K, L and M.
 Find the coordinates of L and M, given that the x-coordinate of M is greater than the x-coordinate of L. [6 marks]



Time to make like x and y in a set of parametric equations, and split...

You quite often get given a sketch of the curve that the parametric equations define. Don't forget that the sketch can be useful for checking your answers — if the curve crosses the x-axis twice, and you've only found one x-coordinate for when y = 0, you know something's gone a bit pear-shaped and you should go back and sort it out, sunshine.

More on Parametric Equations

Now that you've learnt a bit about parametric equations and how great they are...
...here's a page about how to get rid of them.

Rearrange *Parametric Equations to get the* ***Cartesian Equation***

Some parametric equations can be converted into **Cartesian equations**. There are **two main ways** to do this:

To Convert Parametric Equations to a Cartesian Equation:

1. Rearrange one of the equations to make the parameter the subject, then substitute the result into the other equation.

or

2. If your equations involve trig functions, use trig identities (see Section 5) to eliminate the parameter.

You can use the first method to combine the parametrics used in the examples on page 40:

Example: Give the Cartesian equations, in the form $y = f(x)$, of the curves represented by the following pairs of parametric equations:

a) $x = t + 1$ and $y = t^2 - 1$, b) $x = 2t - 3$ and $y = \frac{1}{3t}$, $t \neq 0$.

a) You want the answer in the form $y = f(x)$, so leave y alone for now, and rearrange the equation for x to make t the subject: → $x = t + 1 \Rightarrow t = x - 1$

Now you can eliminate t from the equation for y: → $y = t^2 - 1 \Rightarrow y = (x - 1)^2 - 1 = x^2 - 2x + 1 - 1$

$\Rightarrow$ $y = x^2 - 2x$

b) Rearrange the equation for x to make t the subject: → $x = 2t - 3 \Rightarrow t = \frac{x+3}{2}$

Eliminate t from the equation for y: → $y = \frac{1}{3t} \Rightarrow y = \frac{1}{3\left(\frac{x+3}{2}\right)}$

$\Rightarrow y = \frac{1}{\frac{3(x+3)}{2}} \Rightarrow y = \frac{2}{3x+9}$

If there are ***Trig Functions****... use* ***Trig Identities***

Things get a little trickier when the likes of sin and cos decide to put in an appearance:

Example: A curve has parametric equations

$$x = 1 + \sin\theta, \quad y = 1 - \cos 2\theta$$

Give the Cartesian equation of the curve in the form $y = f(x)$.

If you try to make θ the subject of these equations, things will just get messy. The trick is to find a way to get both x and y in terms of the same trig function. You can get $\sin\theta$ into the equation for y using the identity $\cos 2\theta \equiv 1 - 2\sin^2\theta$:

$$y = 1 - \cos 2\theta$$
$$= 1 - (1 - 2\sin^2\theta) = 2\sin^2\theta$$

Rearranging the equation for x gives:

$\sin\theta = x - 1$, so $y = 2\sin^2\theta$

$\Rightarrow y = 2(x - 1)^2$

$\Rightarrow$ $y = 2x^2 - 4x + 2$

The equation is only valid for $0 \leq x \leq 2$, due to the range of $\sin\theta$ (i.e. $-1 \leq \sin\theta \leq 1$).

If one of the parametric equations includes $\cos 2\theta$ or $\sin 2\theta$, that's probably the one you need to substitute — so make sure you know the double angle formulas (see p.71).

Trigmund, Trigby, Trigor, Triguel and Trigourney tried to conceal their identities.

More on Parametric Equations

Parametric Equations are used in **Modelling**

Sometimes it makes sense to use parametric equations to model a real-life situation. For example, if you're modelling the movement of an object, you can use the parameter t to show how its x- and y-coordinates change with time.

Example: A flying disc is thrown from the point (0, 0). After t seconds, it has travelled x m horizontally and y m vertically, modelled by the parametric equations $x = t^2 + 2t$ and $y = 6t - t^2$ $(0 \le t \le 6)$.

a) Find the x- and y- values of the position of the disc after 2.5 seconds.

b) After how many seconds does the disc reach a height of 5 metres?

c) How far above its starting point is the disc when it reaches the point $x = 24$ m?

a) **Substitute** $t = 2.5$ into the equations for x and y:

$t = 2.5 \Rightarrow x = 2.5^2 + 2 \times 2.5 = 6.25 + 5 =$ **11.25**

$\Rightarrow y = 6 \times 2.5 - 2.5^2 = 15 - 6.25 =$ **8.75**

b) Now you want the value of t when $y = 5$:

$5 = 6t - t^2 \Rightarrow t^2 - 6t + 5 = 0$

$\Rightarrow (t - 1)(t - 5) = 0$

$\Rightarrow$ **$t = 1$ s** and **$t = 5$ s**

c) Use the equation for x to find t first:

$24 = t^2 + 2t \Rightarrow t^2 + 2t - 24 = 0$

$\Rightarrow (t + 6)(t - 4) = 0$

$\Rightarrow t = -6$ or $t = 4$, but $0 \le t \le 6$, so $t = 4$ s

Then use it in the other equation to find y:

$t = 4 \Rightarrow y = 6 \times 4 - 4^2 = 24 - 16 =$ **8 m**

In the example above, the parametric equations are only **valid** for certain values of t.
The **domain** of the parameter is sometimes **restricted**, for example:

1) If the equations are modelling a situation where only certain values of the parameter make sense — e.g. to avoid negative values for time or height.
2) To avoid repeating values of x and y — e.g. if the parametric equations involve $\cos\theta$ or $\sin\theta$, the value of θ might be restricted to $-\pi \le \theta \le \pi$ or $0 \le \theta \le 2\pi$.
3) To avoid dividing by zero — e.g. in the examples on pages 40 and 42, where $y = \frac{1}{3t}$, t is restricted to $t \ne 0$ so you don't end up with zero on the bottom of the fraction.

Restrictions on the value of the parameter will, in turn, usually restrict the values of x and y. The range of a function can also restrict x and y — e.g. $-1 \le \sin\theta \le 1$.

Practice Questions

Q1 Find the Cartesian equation of the curve defined by the parametric equations $x = \frac{t+2}{3}$ and $y = 2t^2 - 3$.

Q2 The parametric equations of a curve are $x = 2\sin\theta$ and $y = \cos^2\theta + 4$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.
a) What is the Cartesian equation of the curve?
b) What restrictions are there on the values of x for this curve?

Q3 Curve C is defined by the parametric equations $x = \frac{\sin\theta}{3}$ and $y = 3 + 2\cos 2\theta$. Find the Cartesian equation of C.

Exam Questions

Q1 A football is kicked from the point (0, 0) on the surface of a flat field. After t seconds, it is modelled as having travelled x m horizontally and y m vertically from its starting point, where $x = 15t$ and $y = 20t - 5t^2$.

a) Find the Cartesian equation for the movement of the ball. [2 marks]

b) How far is the ball from its starting point after 2 seconds? [2 marks]

c) The model is valid for $0 \le t \le 4$. Suggest why it is not valid for values of t outside this range. [2 marks]

Q2 The parametric equations of curve C are $x = 3 + 4\sin\theta$, $y = \frac{1 + \cos 2\theta}{3}$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

a) Show that the Cartesian equation of C can be written $y = \frac{-x^2 + 6x + 7}{24}$. [4 marks]

b) State the domain of values of x for the curve C. [1 mark]

Cartesy peasy, lemon squeezy...

To get a Cartesian equation, you don't always need to get the parameter on its own. If you can rearrange both equations to have the same function of the parameter on one side (e.g. t^3 or $\sin\theta$), then the other sides are equal. Remember, a Cartesian equation just has to link x and y — it doesn't always have to be in the form $y = f(x)$.